# BEAM SIZE ENHANCEMENT DUE TO THE PRESENCE OF SEXTUPOLE MAGNETS IN A RING* 

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A perturbation method ${ }^{1}$ using the Green's function of the Fokker-Planck equation is applied to evaluate an enhancement of the vertical beam size of a bunch due to the presence in a storage ring of sextupolar magnetic fields. The size of the bunch is presented as the power series in sextupole strength parameter.

Consider the beam size in function of the machine tune. The curve will correctly describe the size enhancement only if tune is not too close to a resonance. In immediate vicinity of the resonance the beam size will deviate from correct value, since here breaks the assumption put into the base of the calculation. Namely, the perturbed distribution function will deviate strongly from its unperturbed value. The perturbation theory cannot be used near the resonance and other methods are needed to treat the problem. On the other hand, far from resonance the assumption is valid and the method gives an estimate of the perturbed beam size. A numerical example of the beam enhancement due to the presence of sextupole magnets in PEP storage ring is presented. Discussion of this in more detail can be found in Ref. 2.

To describe a particle motion in a storage ring we use the Courant-Snyder variables ${ }^{3} u, \phi\left(u^{\prime} \equiv d u / d \phi\right)$ for the horizontal and $v, \theta\left(v^{\prime} \equiv d v / d \theta\right)$ for the vertical planes respectively. The sudden change in the particle velocity by a passage through a nonlinear magnet ('kick') in these variables is connected to the kick in variables $x$ and $y$ by the following relationship [cf. expressions (5.8) and (5.10) of Ref. 1]:

$$
\begin{gather*}
\tilde{F}_{x}(u, v)=\nu \sqrt{\beta_{x}} F_{x}|x(u), y(v)|, \text { and }  \tag{1}\\
\tilde{F}_{y}(u, v)=\tau \sqrt{\beta_{y}} F_{y}|x(u), y(v)|, \tag{2}
\end{gather*}
$$

where $\nu$ and $\tau$ are horizontal and vertical tunes of the machine correspondingly.

For a sextupole $F_{x}=S\left(x^{2}-y^{2}\right), F_{y}=-2 S x y$, where $S$ is the integrated strength of the magnet. We get in the variables $u, v$;

$$
\begin{gather*}
\tilde{F}_{x}=\nu\left(S^{(1)} u^{2}-S^{(2)} v^{2}\right) \quad \text { and }  \tag{3}\\
\tilde{F}_{y}=-2 \tau S^{(2)} u v, \quad \text { where }  \tag{4}\\
S^{(1)}=S \beta_{x}^{3 / 2} \quad \text { and }  \tag{5}\\
S^{(2)}=S \beta_{x}^{1 / 2} \beta_{y} \tag{6}
\end{gather*}
$$

The perturbed distribution function $\psi=\psi_{0}+\psi_{1}+\psi_{2}$, where $\psi_{1}$ and $\psi_{2}$ are the first and the second order corrections for the distribution function, ${ }^{1}$ allows us to calculate the perturbed vertical beam emittance $E_{y}$ :

$$
\begin{equation*}
E_{y}=\int d V \psi(V)\left(v^{2}+v^{12} / \tau^{2}\right) / 2 \tag{7}
\end{equation*}
$$

Since the distribution function $\psi$ is found in the form of a series expansion, the vertical beam emittance $E_{y}$ is also an expansion. The zeroth order term of this series is the unperturbed beam emittance $\epsilon_{y}$. It is easy to see that due to symmetry of the sextupole field, the first order term in $E_{y}$ is zero. Hence

$$
\begin{gather*}
\frac{E_{y}}{\epsilon_{y}}=1+\Delta_{2}, \text { where }  \tag{8}\\
\Delta_{2}=\frac{1}{2 \epsilon_{y}} \sum_{k<\ell} \int d V\left(v^{2}+v^{\prime 2} / \tau^{2}\right) \int d V_{1} G\left(V, V_{1}, s_{\ell k}\right) \\
\times \sum_{m<k}\left(\tilde{F}_{x} \frac{\partial}{\partial u^{\prime}}+\tilde{F}_{y} \frac{\partial}{\partial v^{\prime}}\right)_{V_{1}} \int d V_{0} G\left(V_{1}, V_{0}, s_{k m}\right) \\
 \tag{9}\\
\times\left(\tilde{F}_{x} \frac{\partial \psi_{0}}{\partial u^{\prime}}+\tilde{F}_{y} \frac{\partial \psi_{0}}{\partial v^{\prime}}\right)_{V_{0}} .
\end{gather*}
$$

Here $V=\left(u, u^{\prime}, v, v^{\prime}\right)$ and $V_{0}=\left(u_{0}, u_{0}^{\prime}, v_{0}, v_{0}^{\prime}\right)$, are points in a four-dimensional phase space of the transverse motion,

$$
\begin{equation*}
\psi_{0}=\exp \left\{-\frac{u^{2}}{2 \epsilon_{x}}-\frac{u^{\prime 2}}{2 \epsilon_{x} \nu^{2}}-\frac{v^{2}}{2 \epsilon_{y}}-\frac{v^{\prime 2}}{2 \epsilon_{y} \tau^{2}}\right\} /(2 \pi)^{2} \epsilon_{x} \nu \epsilon_{y} \tau \tag{10}
\end{equation*}
$$

and

$$
\begin{align*}
G\left(V, V_{0}, s_{k m}\right)= & G_{u}\left(u, u^{\prime}, \phi_{k} \mid u_{0}, u_{0}^{\prime}, \phi_{m}\right) \\
& \times G_{v}\left(v, v^{\prime}, \theta_{k} \mid v_{0}, v_{0}^{\prime}, \theta_{m}\right) \tag{11}
\end{align*}
$$

is the Green's function as it is discussed in Ref. 1. It is simplier to perform the space integrations first over $V$, then over $V_{1}$ and at last over $V_{0}$. The integral of $v^{2}$ over $V$ is the second Green's function moment $P_{2}=p_{0}+p_{1}^{2} v_{1}^{2}+p_{2}^{2} v_{1}^{\prime 2}+2 p_{3} v_{1} v_{1}^{\prime}$, which has been evaluated in Appendix B of part $I^{1}$ [see formulae (B.12) through (B.15) for coefficients $\left.p_{i}(\theta)\right]$. The second Green's function moment (of $v^{\prime 2}$ ) $Q_{2}=q_{0}+q_{1}^{2} v_{1}^{2}+q_{2}^{2} v_{1}^{\prime 2}+2 q_{3} v_{1} v_{1}^{\prime}$ is found in Appendix B of Part $\Pi^{4}$ [see formulae (B.10) through (B.13) for coefficients $q_{i}(\theta)$. Since neither $P_{2}$, nor $Q_{2}$ depend on $u_{1}^{\prime}$, only the term containing $\tilde{F}_{y}$ contributes to the integral over $V_{1}$. In addition to this, only terms in $P_{2}$ and $Q_{2}$ which depend on $v_{1}^{\prime}$ contribute to the value of the integral. Since $\tilde{F}_{y}\left(u_{1}, v_{1}\right)$ is proportional to $u_{1} v_{1}$ [compare expression (4)], the integral over $V_{1}$ is the product of the first moment of $G_{u}$ ( $P_{1}$ in the notation of Ref. 1) and the sum of two second moments of $G_{v}$ ( $\dot{P}_{2}$ and $P_{2}$ in the same notation):

$$
\begin{align*}
& \quad P_{1}=\tilde{p}_{1} u_{0}+\tilde{p}_{2} u_{0}^{\prime}, \text { where }  \tag{12}\\
& \tilde{p}_{1}=e^{-\alpha \phi}\left(\cos \nu \phi+\frac{\alpha}{\nu} \sin \nu \phi\right),  \tag{13}\\
& \tilde{p}_{2}=e^{-\alpha \phi} \sin \nu \phi / \nu \tag{14}
\end{align*}
$$

Here $\phi$ stands for $\phi_{k}-\phi_{m}$. Further,

$$
\begin{equation*}
P_{2}=p_{0}+p_{1}^{2} v_{0}^{2}+p_{2}^{2} v_{0}^{2}+2 p_{3} v_{0} v_{0}^{\prime} \quad \text { and } \tag{15}
\end{equation*}
$$

$$
\begin{equation*}
\dot{P}_{2}=r_{0}+r_{1} v_{0}^{2}+r_{2} v_{0}^{\prime 2}+2 r_{3} v_{0} v_{0}^{\prime} \tag{16}
\end{equation*}
$$

Expressions for coefficients $p_{i}(\theta)$ and $r_{i}(\theta)$, where $\theta=\theta_{k}-$ $\theta_{m}$, may be found in Appendix B, Part I. ${ }^{1}$

The last integration in (3.4) over $V_{0}$ is very simple. The result is:

$$
\begin{align*}
\Delta_{2}= & 2 \sum_{k<\ell} S_{k}^{(2)}\left\{\tau ^ { 2 } f _ { 2 } \sum _ { m < k } \left[2 \epsilon_{x} S_{m}^{(2)} e^{-\alpha \phi-2 \delta \theta}\right.\right. \\
& \times \cos 2 \tau \theta\left(\cos \nu \phi+\frac{\alpha}{\nu} \sin \nu \phi\right) \\
& \left.-\epsilon_{y} S_{m}^{(2)} e^{-\alpha \phi-2 \delta \theta} \sin \nu \phi\left(\sin 2 \tau \theta+\frac{\delta}{\tau}-\frac{\delta}{\tau} \cos 2 \pi \theta\right)\right] \\
& +\tau f_{3} \sum_{m<k}\left[\left(\epsilon_{y} S_{m}^{(2)}-\epsilon_{x} S_{m}^{(1)}\right) e^{-\alpha \phi} \sin \nu \phi\right. \\
& +\epsilon_{y} S_{m}^{(2)} e^{-\alpha \phi-2 \delta \theta} \sin \nu \phi\left(1+\cos 2 \tau \theta+2 \frac{\delta}{\tau} \sin 2 \pi \theta\right) \\
& +2 \epsilon_{x} S_{m}^{(2)} e^{-\alpha \phi-2 \delta \theta}\left(\cos \nu \phi+\frac{\alpha}{\nu} \sin \nu \phi\right) \\
& \left.\left.\times\left(\sin 2 \tau \theta+\frac{\delta}{\tau}-\frac{\delta}{\tau} \cos 2 \tau \theta\right)\right]\right\} . \tag{17}
\end{align*}
$$

Here $\theta$ denotes $\theta_{k}-\theta_{m}, \phi=\phi_{k}-\phi_{m}$, and

$$
\begin{align*}
& f_{2}=e^{-2 \delta \bar{\theta}}\left(1-\frac{\delta}{\tau} \sin 2 \tau \tilde{\theta}\right) / \tau^{2}  \tag{18}\\
& f_{3}=\delta e^{-2 \delta \tilde{\theta}}(1-\cos 2 \tau \tilde{\theta}) / \tau^{2} \tag{19}
\end{align*}
$$

where $\tilde{\theta}$ stands for $\theta_{\ell}-\theta_{k}$.
The subscript ( $k$ or $m$ ) in the notation of sextupole magnets numbers them in the order in which they are seen by a bunch.

Consider now the general case of $n$ arbitrarily positioned sextupoles in the ring. Taking as an example the typical sum over $m$ of cosine terms in expression (17), we present it in the following form:

$$
\begin{align*}
R_{1}^{+}= & \left.\sum_{m>k} S_{m}^{(2)} \exp \mid-\alpha\left(\phi_{m}-\phi_{k}\right)-2 \delta\left(\theta_{m}-\theta_{k}\right)\right] \\
& \times \cos \left[2 \tau\left(\theta_{m}-\theta_{k}\right)+\nu\left(\phi_{m}-\phi_{k}\right)\right] \\
& =S_{1}^{(2)}\left[\frac{1}{2}+\sum_{m=1}^{\infty} \exp (-2 \pi \alpha m-4 \pi \delta m) \cos (4 \pi m m+2 \pi \nu m)\right] \\
& +S_{2}^{(2)} \exp \left[-\alpha\left(\phi_{2}-\phi_{1}\right)-2 \delta\left(\theta_{2}-\theta_{1}\right)\right] \\
& \times \sum_{m=0}^{\infty} \exp (-2 \pi \alpha m-4 \pi \delta m) \\
& \times \cos \left[4 \pi \pi m+2 \pi \nu m+2 \pi\left(\theta_{2}-\theta_{1}\right)+\nu\left(\phi_{2}-\phi_{1}\right)\right]+\ldots \\
& +S_{n}^{(2)} \exp \left[-\alpha\left(\phi_{n}-\phi_{1}\right)-2 \delta\left(\theta_{n}-\theta_{1}\right)\right] \\
& \times \sum_{m=0}^{\infty} \exp (-2 \pi \alpha m-4 \pi \delta m)  \tag{20}\\
& \times \cos \left[4 \pi m m+2 \pi \nu m+2 \pi\left(\theta_{n}-\theta_{1}\right)+\nu\left(\phi_{n}-\phi_{1}\right)\right]
\end{align*}
$$

Here quite arbitrarily the starting sextupole is numbered as $S_{1}$. In the first term of expression (20) only half pulse at "time" zero is taken into account in accord with the summation rule developed in Ref. 4. Performing some algebra we get:

$$
\begin{align*}
R_{1}^{+}= & \frac{\pi(2 \delta+\alpha)}{1-\cos 2 \pi(2 \tau+\nu)} \sum_{i=1}^{n} S_{i}^{(2)} \\
& \times \cos \left[2 \tau\left(\theta_{i}-\theta_{1}\right)+\nu\left(\phi_{i}-\phi_{1}\right)\right] \\
& +\frac{1}{2 \sin \pi(2 \tau+\nu)} \sum_{i=2}^{n} S_{i}^{(2)}  \tag{21}\\
& \times \exp \left[-\alpha\left(\phi_{i}-\phi_{1}\right)-2 \delta\left(\theta_{i}-\theta_{1}\right)\right] \\
& \times \sin \left[\pi(2 \tau+\nu)-2 \tau\left(\theta_{i}-\theta_{1}\right)-\nu\left(\phi_{i}-\phi_{1}\right)\right] .
\end{align*}
$$

There are in general $n$ different sums $R_{j}^{+}, j=1,2, \ldots, n$, similar to $R_{1}^{+}$. They appear, when one consider similar sums starting from the sextupole, positioned at $j$-th place in the ring. Any sum evaluated for the sextupole shifted by $n+1$ positions from the $j$-th one is equal to the $j$-th sum:

$$
\begin{equation*}
R_{j+n}^{+}=R_{j}^{+} \tag{22}
\end{equation*}
$$

Exponents in each of the second sum in expression (21) are very close to 1 , since $\alpha$ and $\delta$ are usually small. Hence, the exponential factors may be expanded and all the terms of the order $(\alpha \Delta \phi)^{2}$ and $(\delta \Delta \theta)^{2}$ and higher may be omitted.

Let us introduce notations:

$$
\begin{align*}
A_{j}^{+}= & \sum_{i=1}^{n} S_{j+i-1}^{(2)} \cos \left[2 \tau\left(\theta_{j+i-1}-\theta_{j}\right)+\nu\left(\phi_{j+i-1}-\phi_{j}\right)\right]  \tag{23}\\
B_{j}^{+}= & \sum_{i=2}^{n} S_{j+i-1}^{(2)}  \tag{24}\\
& \times \sin \left[\pi(2 \tau+\nu)-2 \tau\left(\theta_{j+i-1}-\theta_{j}\right)-\nu\left(\phi_{j+i-1}-\phi_{j}\right)\right] \\
C_{j}^{+}= & \sum_{i=1}^{n} S_{j+i-1}^{(2)}\left[2\left(\theta_{j+i-1}-\theta_{j}\right)+\frac{\alpha}{\delta}\left(\phi_{j+i-1}-\phi_{j}\right)\right] \\
& \times \sin \left[\pi(2 \tau+\nu)-2 \tau\left(\theta_{j+i-1}-\theta_{j}\right)-\nu\left(\phi_{j+i-1}-\phi_{j}\right)\right] \\
& \quad(j=1,2, \ldots, n) \tag{25}
\end{align*}
$$

These values have the same periodic property (22) as $R_{j}^{+}$. The latter can be now written as follows:

$$
\begin{align*}
R_{j}^{+}= & \frac{\pi(2 \delta+\alpha)}{2 \sin ^{2} \pi(2 \tau+\nu)} A_{j}^{+}+\frac{1}{2 \sin \pi(2 \tau+\nu)} B_{j}^{+} \\
& -\frac{\delta}{2 \sin \pi(2 \tau+\nu)} C_{j}^{+} . \tag{26}
\end{align*}
$$

It is easy to find now the double sum over $k$ and $m$ of the considered term:

$$
\begin{align*}
& \sum_{k>\ell} S_{k}^{(2)} \exp \left[-2 \delta\left(\theta_{k}-\theta_{\ell}\right)\right] R_{k}^{+} \\
= & \frac{(1+\alpha / 2 \delta)}{4 \sin ^{2} \pi(2 \tau+\nu)} a^{+}-\frac{1}{8 \pi \sin \pi(2 r+\nu)} c^{+} \tag{27}
\end{align*}
$$

where a new notations are introduced:

$$
\begin{align*}
& a^{+}=\sum_{j=1}^{n} S_{j}^{(2)} A_{j}^{+}  \tag{28}\\
& c^{+}=\sum_{j=1}^{n} S_{j}^{(2)} C_{j}^{+} \tag{29}
\end{align*}
$$

Notice the absence in expression (27) of the term proportional to $b^{+}=\sum_{j=1}^{n} S_{j}^{(2)} B_{j}^{+}$. Indeed, it is easy to check, that this quantity is equal to zero.

Introduce now similar notations for all other types of terms:

$$
\begin{gather*}
a^{-}=\sum_{j=1}^{n} S_{j}^{(2)} A_{j}^{-}  \tag{30}\\
c^{-}=\sum_{j=1}^{n} S_{j}^{(2)} C_{j}^{-} \text {where }  \tag{31}\\
A_{j}^{-}=\sum_{i=1}^{n} S_{j+i-1}^{(2)} \cos \left[2 \tau\left(\theta_{j+i-1}-\theta_{j}\right)-\nu\left(\phi_{j+i-1}-\phi_{j}\right)\right]  \tag{32}\\
C_{j}^{-}=\sum_{i=1}^{n} S_{j+i-1}^{(2)}\left[2\left(\theta_{j+i-1}-\theta_{j}\right)+\frac{\alpha}{\delta}\left(\phi_{j+i-1}-\phi_{j}\right)\right] \\
\left.\times \sin \mid \pi(2 \tau-\nu)-2 \tau\left(\theta_{j+i-1}-\theta_{j}\right)+\nu\left(\phi_{j+i-1}-\phi_{j}\right)\right]  \tag{33}\\
a_{1,2}^{o}=\sum_{j=1}^{n} S_{j}^{(2)} A_{j}^{(1,2)}, \quad \text { where }  \tag{34}\\
A_{j}^{(1,2)}=\sum_{i=1}^{n} S_{j+i-1}^{(1,2)} \cos \nu\left(\phi_{j}-\phi_{j+i-1}\right) \tag{35}
\end{gather*}
$$

The terms of $c_{1,2}^{o}$ types are of higher order of magnitude in $\alpha$ and do not enter the final result.

In a similar way one can calculate also the sine type terms in expression (17). In this case one needs to introduce quantities

$$
\begin{equation*}
d^{ \pm}=\sum_{j=1}^{n} S_{j}^{(2)} D_{j}^{ \pm} \quad \text { and } \tag{36}
\end{equation*}
$$

$$
\begin{align*}
& \qquad d_{1,2}^{o}=\sum_{j=1}^{n} S_{j}^{(2)} D_{j}^{(1,2)}, \text { where } \\
& D_{j}^{ \pm}=\sum_{i=1}^{n} S_{j+i-1}^{(2)} \\
& \times \cos \left[\pi(2 \tau \pm \nu)-2 \tau\left(\theta_{j+i-1}-\theta_{j}\right) \mp \nu\left(\phi_{j+i-1}-\phi_{j}\right)\right] \tag{38}
\end{align*}
$$

and

$$
\begin{equation*}
D_{j}^{(1,2)}=\sum_{i=2}^{z} S_{j+i-1}^{(1,2)} \cos \left[\pi \nu-\nu\left(\phi_{j+i-1}-\phi_{j}\right)\right] \tag{39}
\end{equation*}
$$

In terms of the quantities $a, c$ and $d$ the final result for $\Delta_{2}$ is:

$$
\begin{align*}
\Delta_{2}= & \frac{1}{4 \pi}\left\{\frac{2 \pi\left(\epsilon_{x}+\epsilon_{y} / 2\right)(1+\alpha / 2 \delta)}{\sin ^{2} \pi(2 \tau+\nu)} a^{+}-\frac{\epsilon_{x}+\epsilon_{y} / 2}{\sin \pi(2 \tau+\nu)} c^{+}\right. \\
& +\frac{2 \pi\left(\epsilon_{x}-\epsilon_{y} / 2\right)(1+\alpha / 2 \delta)}{\sin ^{2} \pi(2 \tau-\nu)} a^{-}-\frac{\epsilon_{x}-\epsilon_{y} / 2}{\sin \pi(2 \tau-\nu)} c^{-} \\
& +\frac{\left(\epsilon_{x}+\epsilon_{x} \frac{\alpha \tau}{\nu \delta}+\epsilon_{y}\right)}{\tau \sin \pi(2 \tau+\nu)} d^{+}+\frac{\left(\epsilon_{x}-\epsilon_{x} \frac{\alpha \tau}{\nu \delta}-\epsilon_{y}\right)}{\tau \sin \pi(2 \tau-\nu)} d^{-} \\
& \left.+\frac{\epsilon_{y} d_{2}^{o}}{\tau \sin \pi \nu}-\frac{\epsilon_{x} d_{1}^{o}}{\tau \sin \pi \nu}\right\} . \tag{40}
\end{align*}
$$

For a ring with $M$ identical superperiods expression (40) is invariant under the following transformation:

$$
\begin{equation*}
\Delta_{2}(\nu, \alpha, \tau, \delta, n)=\Delta_{2}\left(\frac{\nu}{M}, \frac{\alpha}{M}, \frac{\tau}{M} \frac{\delta}{M}, \frac{n}{M}\right) \tag{41}
\end{equation*}
$$

The quantities $a^{ \pm, o}, c^{ \pm}$and $d^{ \pm, o}$ can be named the (sextupole) distribution factors. For a given ring only the distribution factors change, when parameters of the distribution of the sextupole magnets, i.e. their number, strengths and positions in the ring, are changed. In general there are eight distribution factors altogether.

Here is the result of the evaluation of the vertical beam size growth for the current PEP configuration (assuming optimal coupling): $\nu=21.25, \beta_{y}^{*}=0.11 \mathrm{~m}, \tau=18.19, \alpha / \delta=1.0$, $\beta_{x}^{*}=3.0 \mathrm{~m}, \sqrt{\epsilon_{y} / \epsilon_{x}}=0.19, \sigma_{x}=0.58 \mathrm{~mm}$.

The calculation by formula (40) yields:

$$
\begin{equation*}
\frac{\Delta \epsilon_{y}}{\epsilon_{y}}=1.3 \% \tag{42}
\end{equation*}
$$

The beam emittance enhancement of the same magnitude is found also for other points of the tune diagram which are not too close to the resonance lines $2 \tau \pm \nu=$ integer and $\nu=$ integer.

The beam increase $\Delta \epsilon_{y} / \epsilon_{y}$ goes as cube of the beta function value at the sextupole positions. Hence its value might be much larger for a larger size ring. The invariance of the result (42) under the transformation (41) has been confirmed by the numerical calculation.

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