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Note on the Courant and Snyder Invariant

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Abstract

A review is given of the mathematical derivation of the Courant and Snyder invariant which is well-known as the betatron emittance in accelerators or storage rings. It is shown that the existance of such an invariant is a remarkable characteristic of a linear system even for non-periodic motion.

Introduction

Over the past quarter century, a considerable amount of work has been devoted to the study of the time-dependent linear oscillator

$$\ddot{x} + K(s)\,x = 0, \tag{1-1}$$

which represents betatron oscillations in accelerators and storage rings. Courant and Snyder[1] first found that a conserved quantity for Eq.(1-1) is

$$I = \frac{1}{2\beta(s)}\left[\, x^2 + \left(\frac{\dot\beta(s)}{2}x - \beta(s)\dot{x}\right)\right], \tag{1-2}$$

where x(s) satisfies Eq.(1-1) and $\beta(s)$ satisfies the auxiliary equation

$$\frac{1}{2}\beta\ddot\beta - \frac{1}{4}\dot\beta^2 + K(s)\beta^2 = 1. \tag{1-3}$$

Several derivations of the dynamical invariant(1-2) have been given in the literature: The exact invariant was derived by Lewis and Riesenfeld[2] on the assumption of quadratic invariance. Lutzky[3] derived the invariant(1-2) from Noether´s theorem and recently Korsch[4] presented a proof of the dynamical invariance of (1-2) using the method of dynamical algebra. An early discussion about the general interrelation between the differential equation(1-1) and (1-2) can be found in an article by Milne.[5] In addition, a physical meaning of the origin of the invariant was presented by Eliezer and Gray,[6] with the help of auxiliary plane motion.

It is the aim of the present note to review the three different methods for deriving the dynamical invariant (1-2).

Derivation of Invariant

a. Time-Dependent Linear Canonical Transformation

We shall show explicitly that a time-dependent Hamiltonian

$$H(x,p;s) = \frac{1}{2}\left[\, p^2 + K(s)x^2\right], \tag{2-1}$$

can be converted to time-independent form with the help of a time-dependent linear canonical transformation and a change of time scale.

The canonical equations of motion obtained from (2-1) are

$$\dot{x} = \frac{\partial H}{\partial p} = p, \tag{2-2a}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -K(s)x. \tag{2-2b}$$

First we require that the Hamiltonian in Eq.(2-1) is transformed into the form

$$H'(X,P;s) = \frac{f(s)}{2}\left(P^2 + X^2\right), \tag{2-3}$$

with a time-dependent function f(s) which is determined later, by means of the time-dependent linear transformations

$$X = \Lambda_1^1(s)x + \Lambda_2^1(s)p, \tag{2-4a}$$

$$P = \Lambda_1^2(s)x + \Lambda_2^2(s)p. \tag{2-4b}$$

Because we assume the canonical transformation, the time-dependent coefficients $\Lambda_1^1(s)$, $\Lambda_2^1(s)$, $\Lambda_1^2(s)$, and $\Lambda_2^2(s)$ in Eq.(2-4) must satisfy the relation

$$\Lambda_1^1(s)\Lambda_2^2(s) - \Lambda_2^1(s)\Lambda_1^2(s) = 1. \tag{2-5}$$

The canonical equations of motion obtained from Eq.(2-3) are

$$\dot{X} = \frac{\partial H'}{\partial P} = f(s)P, \tag{2-6a}$$

$$\dot{P} = -\frac{\partial H'}{\partial X} = -f(s)X. \tag{2-6b}$$

In order to determine the unknown time-dependent coefficients in (2-3),(2-4), the relations in (2-2),(2-4),and (2-6) are combined in such a manner that the new canonical variables are replaced by the old ones. This is effected by taking the time derivatives of the relations in (2-4), replacing X and P by the expressions (2-6), and then substituting x and p by the quantities given in (2-2). Finally we equate the coefficients of like powers of x, p, x^2, and p^2 from both sides of the equations and obtain the relations among the coefficients

$$\dot\Lambda_1^1 = K(s)\Lambda_2^1 + f(s)\Lambda_1^2, \tag{2-7a}$$

$$\dot\Lambda_2^1 = -\Lambda_1^1 + f(s)\Lambda_2^2, \tag{2-7b}$$

$$\dot\Lambda_1^2 = -f(s)\Lambda_1^1 + K(s)\Lambda_2^2, \tag{2-7c}$$

$$\dot\Lambda_2^2 = -f(s)\Lambda_2^1 - \Lambda_1^2. \tag{2-7d}$$

The coupled equations (2-7) may be solved by the well-known matrix method. However we show a set of particular solutions satisfying Eqs.(2-5),(2-7). Taking Λ_2^1 =0 and replacing Λ_2^2 with $\rho(s)$, it is then trivial to obtain the solution for Λ_1^1 from (2-5). The solution is

$$\Lambda_1^1(s) = \rho^{-1}(s). \tag{2-8}$$

Substituting Λ_2^1 =0, Λ_2^2= $\rho(s)$, and (2-8) into (2-7b), we have

$$f(s) = \rho^{-2}(s). \tag{2-9}$$

Also substituting the time derivative of Λ_1^1, $\Lambda_2^1(=0)$, and (2-9) into (2-7a), we have

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$$\Lambda_1^2 = -\dot{\rho}(s) \tag{2-10}$$

Eq.(2-10) is equivalent to Eq.(2-7d). Next, substituting (2-8), (2-9) and (2-10) into Eq.(2-7a), we obtain the differential equation satisfied by $\rho(s)$,

$$\ddot{\rho} + K(s)\rho = \rho^{-3}. \tag{2-11}$$

If we replace $\rho(s)$ with $\sqrt{\beta(s)}$, the differential equation for the so-called betatron amplitude function $\beta(s)$ will be easily written down

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + K(s)\beta^2 = 1. \tag{2-12}$$

Furthermore, if the change of independent variable

$$\phi(s) = \int^s f(s')ds', \tag{2-13}$$

is made, the Hamiltonian H′ becomes

$$H''(X, P; \phi) = \frac{1}{2}(P^2 + X^2). \tag{2-14}$$

Evidently the new Hamiltonian H" is a constant of motion in the coordinate system of $(X,P;\phi)$. It is apparent that Eq.(2-14) is invariantin the old system $(x,p;s)$;

$$\frac{dH''}{ds} = \frac{dH''}{d\phi}\frac{d\phi}{ds} = f(s)\frac{\partial H''}{\partial\phi} = 0.$$

Next let us show (2-14) as a function of $\beta(s)$, x and $\dot{x}$. Using

(2-8) and (2-10), we write the s - dependent coefficients of $\Lambda_1^1(s), \Lambda_2^1(s), \Lambda_1^2(s)$, and $\Lambda_2^2(s)$ in (2-4a),(2-4b) with the function $\beta(s)$,

$$\Lambda_1^1(s) = \beta^{-\frac{1}{2}}(s), \quad \Lambda_2^1(s) = 0,$$
$$\Lambda_1^2(s) = -\frac{1}{2}\beta^{-\frac{1}{2}}(s)\cdot\dot{\beta}(s), \quad \Lambda_2^2(s) = \beta^{\frac{1}{2}}(s). \tag{2-15}$$

Introducing these values in (2-4a) and (2-4b) and substituting them into (2-14), we obtain the invariant

$$H'' = \frac{1}{2}\left[\left(-\frac{1}{2}\beta^{-\frac{1}{2}}\dot{\beta}x + \beta^{\frac{1}{2}}p\right)^2 + \left(\beta^{-\frac{1}{2}}x\right)^2\right]. \tag{2-16}$$

Setting $p=\dot{x}$ and H"= I in (2-16), we write the dynamical invariant in the form

$$I = \frac{1}{2\beta(s)}\left[x^2 + \left(\frac{\dot{\beta}(s)}{2}x - \beta(s)\dot{x}\right)^2\right]. \tag{2-17}$$

b. Dynamical Algebra

We assume the dynamical invariant in the quadratic form

$$I = \frac{1}{2}\lambda_1(s)p^2 + \lambda_2(s)px + \frac{1}{2}\lambda_3(s)x^2, \tag{2-18}$$

for the harmonic oscillator (2-1). The coefficients $\lambda_n(s)$ are determined by the time evolution equation for a phase-space function

$$\frac{\partial I}{\partial s} + [I, H] = 0, \tag{2-19}$$

where [,] is the Poisson bracket. Substituting (2-1) and (2-18) into (2-19), we obtain the differential equations for $\lambda_n(s)$

$$\dot{\lambda} = M\lambda, \tag{2-20}$$

with $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T$

and

$$M = \begin{pmatrix} 0 & -2 & 0 \\ K(s) & 0 & -1 \\ 0 & 2K(s) & 0 \end{pmatrix}.$$

Here, setting $\lambda_1 = \beta_c(s)$, we find

$$\lambda_2 = -\frac{1}{2}\dot{\beta}_c, \tag{2-21a}$$
$$\dot{\lambda}_3 = -K(s)\dot{\beta}_c, \tag{2-21b}$$
$$\lambda_3 = \frac{1}{2}\ddot{\beta}_c + K(s)\beta_c. \tag{2-21c}$$

Equating the derivative of (2-21c) with (2-21b), we finally obtain

$$\dddot{\beta}_c + 4K(s)\dot{\beta}_c + 2\dot{K}(s)\beta_c = 0, \tag{2-22}$$

which has the integral

$$\frac{1}{2}\beta_c\ddot{\beta}_c - \frac{1}{4}\dot{\beta}_c^2 + K(s)\beta_c = C. \tag{2-23}$$

The solution of (2-23) determines the $\lambda_n(s)$ and the dynamical invariant (2-18) is therefore expressed in the form

$$I = \frac{1}{2\beta(s)}\left[x^2 + \left(\frac{\dot{\beta}(s)}{2}x - \beta(s)p\right)^2\right]. \tag{2-24}$$

Here the arbitrariness implied by the constant C is removed after some manipulations.

The dynamical invariant for an off-momentum particle has been also obtained by the present author,[7] based on dynamical algebra.

c. Noether's Theorem

The formulation of Noether's theorem presented in this section has been given by Lutzky. If the transformation

$$G = \xi(x,s)\frac{\partial}{\partial s} + \eta(x,s)\frac{\partial}{\partial x}$$

leaves the action integral $\int L(x,\dot{x};s)ds$ invariant,

$$\xi\frac{\partial L}{\partial s} + \eta\frac{\partial L}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi})\frac{\partial L}{\partial \dot{x}} + \dot{\xi}L = \dot{f}, \tag{2-25}$$

where f=f(s,t), and

$$\dot{\xi} = \frac{\partial\xi}{\partial s} + \dot{x}\frac{\partial\xi}{\partial x}, \quad \dot{\eta} = \frac{\partial\eta}{\partial s} + \dot{x}\frac{\partial\eta}{\partial x}, \quad \dot{f} = \frac{\partial f}{\partial s} + \dot{x}\frac{\partial f}{\partial x},$$

then a constant of the motion for the system is given by

$$\Phi = (\xi\dot{x} - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + f. \tag{2-26}$$

The Lagrangian $L = \frac{1}{2}(\dot{x}^2 - K(s)\cdot x^2)$ gives the equation of motion (1-1); using this lagrangian in (2-25) and equating coefficients of powers of $\dot{x}$ to zero, we obtain a set of equations for ξ, η, f

$$\frac{\partial\xi}{\partial x} = 0, \tag{2-27a}$$
$$\frac{\partial\eta}{\partial x} - \frac{1}{2}\frac{\partial\xi}{\partial s} = 0, \tag{2-27b}$$
$$\frac{\partial\eta}{\partial s} - \frac{1}{2}K(s)\frac{\partial\xi}{\partial x} - \frac{\partial f}{\partial x} = 0, \tag{2-27c}$$
$$-\frac{1}{2}\xi\dot{K}(s)x^2 - \eta K(s)x - \frac{1}{2}K(s)\frac{\partial\xi}{\partial s}x^2 - \frac{\partial f}{\partial s} = 0. \tag{2-27d}$$

Eq.(2-27a) implies that ξ is a function of s alone. From (2-27b) and (2-27c), we obtain the results

$$\eta(x,s) = \frac{1}{2}\dot{\xi}x + \psi(s), \tag{2-28}$$
$$f(x,s) = \frac{1}{4}\ddot{\xi}x^2 + \dot{\psi}(s)x + C(s), \tag{2-29}$$

where $\ddot{\psi}(s) + K(s)\cdot\psi(s)=0$ and $C(s)$ is an arbitrary function of s alone. Choosing $C(s)=0$, $\psi(s)=0$ and substituting (2-28),(2-29) into (2-27d), we find

$$\dddot{\xi} + 4K(s)\dot{\xi} + 2\dot{K}(s)\xi = 0. \qquad (2\text{-}30)$$

Eq.(2-30) has the integral

$$\frac{1}{2}\xi\ddot{\xi} - \frac{1}{4}\dot{\xi}^2 + K(s)\xi^2 = C, \qquad (2\text{-}31)$$

where C is an integration constant. Replacing ξ with $\beta_c(s)$ in (2-28), (2-29), and (2-31), we have

$$h(x,s) = \frac{1}{2}\dot{\beta}_c(s)x, \qquad (2\text{-}32a)$$

$$f(x,s) = \frac{1}{4}\ddot{\beta}_c(s)x^2, \qquad (2\text{-}32b)$$

$$\frac{1}{2}\beta_c\ddot{\beta}_c - \frac{1}{4}\dot{\beta}_c^2 + K(s)\beta_c^2 = C. \qquad (2\text{-}32c)$$

Further using (2-32c), we obtain

$$f(x,s) = \frac{1}{2}\left[\frac{C}{\beta_c} + \frac{\dot{\beta}_c^2}{4\beta_c} - K(s)\beta_c\right]x^2. \qquad (2\text{-}33)$$

Finally setting $\xi = \beta_c$ in (2-26) and substituting (2-32a),(2-33) into (2-26), we write the invariant

$$\Phi = \frac{1}{2\beta(s)}\left[x^2 + \left(\frac{\dot{\beta}(s)}{2}x - \beta(s)\dot{x}\right)^2\right] \qquad (2\text{-}34)$$

Here the arbitrariness by the presence of the constant C is removed in the same way as in the previous subsection.

Remark

As a natural extension of the present discussion, an explicit expression of the Courant and Snyder invariant for a linear coupled betatron oscillation produced by skew quadrupole components can be obtained. The invariant must be accompanied by new auxiliary conditions. These conditions are satisfied by perturbed betatron amplitude functions. In particular, a dynamical invariant will be presented elsewhere for a time-dependent weak coupled harmonic oscillator.[8]

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