ACHROMATICITY VS. ISOCHRONICITY

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Summary

An achromatic system is one where the transfer matrix elements for the transverse coordinates do not depend on momentum. An isochronous system is one where the transit time of a trajectory through the system does not depend on the initial coordinates. It is well known that a first-order achromatic system is also isochronous, except for pure momentum dependence. The converse is also true. This result is entended to higher orders. Conditions are found so that for a system whose chromatic terms all vanish up to a certain order the transit time will be independent of the transverse coordinates up the same order. Under the same conditions, the converse will also be true.

Introduction

The location in phase space of a particle passing through a magnetic optical system is usually specified with respect to a reference trajectory. The three spatial coordinates of a beam particle are the two transverse coordinates x and y and the distance s along the reference trajectory. Corresponding to these three coordinates are the three conjugate momenta p_x, p_y, and p_s. In a field-free region, these three components reduce to the three Cartesian components of the mechanical momentum.

In practice, the two transverse momenta are replaced by the two direction tangents or "angles" x' and y'. The longitudinal position s is replaced by the longitudinal separation ℓ from the reference particle. The sixth component is the fractional deviation δ of the momentum from the reference momentum. The position and momentum of a particle at any point in a beam line can be expressed in terms of a six-component vector X, where

$$X = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{pmatrix} \qquad (1)$$

The components of the vector at any point in the beam line can be expressed as functions of the components of the initial ray vector. Retaining only linear terms yields

$$X_1 = RX_o \qquad (2)$$

Defining matrix elements T, U, etc. with several indices, and summing those indices also, the expansion may be extended to higher orders, so that

$$X_1 = RX_o + TX_oX_o + UX_oX_oX_o + \cdots \qquad (3)$$

The matrix elements of R are referred to as being of first order; T is second order; U is third, etc.

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We may now define precisely what we mean by the terms achromatic and isochronous. A system is achromatic to a certain order if the final matrix elements for the transverse coordinates x, x', y, and y' have no dependence on δ to that order. There may still exist nonzero geometric matrix elements, where the matrix element multiplies products of initial transverse coordinates. A system is isochronous to a certain order if the transit time difference has no dependence on the initial transverse coordinates to that order. For an ultrarelativistic beam, equal transit time is the same as equal path length. For lower energies, velocity differences must be included. The differences between the two appear first in second order. Our result applies strictly only to transit time differences. There may still be a dependence on powers of δ unmixed with other coordinates.

First Order

We may use a bra and ket notation to indicate any particular matrix element. The columns of the first-order transfer matrix are also known as the characteristic rays, so that

$$\begin{aligned} (x|x_o) &= c_x(s) & (x'|x_o) &= c_x'(s) \qquad (4)\\ (x|x_o') &= s_x(s) & (x'|x_o') &= s_x'(s) \\ (x|\delta) &= d_x(s) & (x'|\delta) &= d_x'(s) \\ (y|y_o) &= c_y(s) & (y'|y_o) &= c_y'(s) \\ (y|y_o') &= s_y(s) & (y'|y_o') &= s_y'(s) \end{aligned}$$

The dispersion may be given in terms of the transverse characteristic rays, and the angle α of the bend magnets as

$$\begin{aligned} d_x(s) &= s_x(s)\int c_x(s)\ d\alpha - c_x(s)\int s_x(s)\ d\alpha \qquad (5)\\ d_x'(s) &= s_x'(s)\int c_x(s)\ d\alpha - c_x'(s)\int s_x(s)\ d\alpha \end{aligned}$$

The longitudinal separation is given as

$$\ell = x_o\int c_x(s)d\alpha + x_o'\int s_x(s)d\alpha + \delta\int d_x(s)d\alpha \qquad (6)$$

From equations (5) and (6), we see that the stated theorem connecting achromaticity and isochronicity is true in first order. The relation between the coefficients can be written as

$$\frac{\partial x}{\partial\delta} = \frac{\partial x}{\partial x_o'}\frac{\partial\ell}{\partial x_o} - \frac{\partial x}{\partial x_o}\frac{\partial\ell}{\partial x_o'} \qquad (7)$$

$$\frac{\partial x'}{\partial\delta} = \frac{\partial x'}{\partial x_o'}\frac{\partial\ell}{\partial x_o} - \frac{\partial x'}{\partial x_o}\frac{\partial\ell}{\partial x_o'}$$

This form is highly suggestive of how the result may be extended to higher orders.

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Relation to Canonical Variables

In a charged particle beam or spectrometer, one is interested in the behavior at the end as a function of the trajectory coordinates at the beginning. Since the time of transit may depend on the initial coordinates, it is more convenient to parameterize the equations of motion in terms of a coordinate which has a unique value at the end of the system. For this purpose, we use the distance s along the reference trajectory.

Following Dragt[1], we may then regard the transit time and the Hamiltonian as conjugate variables. A new Hamiltonian may be derived as the momentum conjugate to the distance along the central trajectory. It will be a function of three coordinates and their conjugate momenta. The three coordinates are the two transverse coordinates x and y, and the transit time t. The conjugate momenta are the two transverse momenta and the original Hamiltonian. In a static magnetic system, the value of the Hamiltonian is equal to the kinetic energy of the particle.

The new dependent variables are completely canonical and satisfy Hamilton's equations of motion. The Hamiltonian and the transit time now have no special status and are just one of three sets of conjugate variables. By a canonical transformation, they can be replaced by differences, τ and ε, with respect to the reference trajectory.

Poisson Brackets

The complete set of variables for a particle trajectory is now x, p_x, y, p_y, τ, and ε. The symbols τ and ε indicate respectively the time separation of an individual trajectory and the reference particle, and the difference in energy for the same two particles. Since the variables are canonical and satisfy Hamilton's equations, they also satisfy the fundamental Poisson brackets

$$[q_i,q_j] = 0 \qquad (8)$$

$$[p_i,p_j] = 0$$

$$[q_i,p_j] = \delta_{ij}$$

Here we are interested in the Poisson brackets between τ and the transverse variables. The brackets involving τ and x and those involving τ and y are between different coordinates. Those between τ and p_x and between τ and p_y are between a coordinate and a non-conjugate momentum. All equal zero. A further simplification occurs since we are working with a static system. The transverse coordinates then have no explicit dependence on initial time difference, and the derivative of final time difference with respect to initial time difference is unity.

The result can then be written as

$$\frac{\partial}{\partial \varepsilon}\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \frac{\partial(x,p_x,y,p_y)}{\partial(x_o,p_{xo},y_o,p_{yo})}\,\frac{\partial \tau}{\partial(p_{xo},x_o,p_{yo},y_o)} \qquad (9)$$

We can define two vectors T_c and L_c, and thereby write equation (9) in the form

$$T_c = ML_c \qquad (10)$$

The vector T_c represents the left side of equation (9), which is the derivative of the transverse canonical variables with respect to ε,. The vector L_c represents the derivatives of τ with respect to the initial canonical transverse variables. The Matrix M is the local linearization in canonical variables of the mapping of the original space of transverse coordinates and momenta into the final space of the same variables. By Liouville's theorem, the phase space volume is conserved and the determinant of M is equal to one.

Relation to Transport Variables

Using the chain rule, the vectors T_c and L_c, can be expressed in terms of derivatives with respect to the transverse coordinates given in equation (1). The procedure is more straightforward for the longitudinal vector L_c, so we consider that first.

We define a new vector L, which contains the derivatives of τ with respect to the variables x_o, x'_o, y_o, and y'_o. We now have

$$L_c = NL \qquad (11)$$

where N is a four-by-four matrix. In lowest order N takes a particularly simple form

$$N \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/p_o & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/p_o \end{pmatrix} \qquad (12)$$

The matrix is nonsingular in lowest order, which is its exact form when evaluated at the origin. By continuity, it is then nonsingular in an open region containing the origin.

The connection between L, the dependence of time difference on initial transverse coordinates, and T_c, the dependence of the final canonical transverse coordinates on ε, is then made easily. From equations (10) and (11), we have

$$T_c = MNL \qquad (13)$$

However, what we want is the dependence of final transport variables x, x', y, and y' on initial ε. From the standpoint of canonical variables a drift space has chromatic dependence. If we hold the initial p_x and p_y constant, and vary the energy, the final transverse position will be affected. This occurs because the longitudinal momentum p_t is changed and therefore the angles x' and y' are affected. In studying achromaticity, we are interested in the dependence on energy when the initial angles x' and y' are held fixed.

To convert to transport variables, we must use the chain rule at both the initial and final points. We use it at the final point simply to transform to the desired variables. We use it at the initial point because a derivative with respeect to ε holding p_{xo} fixed is different from a derivative holding x' fixed. We therefore have three sets of partial derivatives. The first is the set of derivatives of the final canonical variables with respect to the final transverse transport variables. Then we have the transfer functions giving the transformation of the transport variables by the beam line. Finally, we must express the initial transport variables in terms of canonical variables.

The energy difference ε is unchanged by all these transformations, but the functional dependence on it is not. In order to be explicit about the functional dependence, we add a subscript to the variable ε, indicating the beginning or the end of the system. Derivatives with respect to canonical sets of coordinates are indicated by a subscript c. For the horizontal components of T_c, we then have

$$\frac{\partial x_1}{\partial \varepsilon_o}\bigg|_c = \frac{\partial x_1}{\partial x'_o}\frac{\partial x'_o}{\partial \varepsilon_o} + \frac{\partial x_1}{\partial y'_o}\frac{\partial y'_o}{\partial \varepsilon_o} + \frac{\partial x_1}{\partial \varepsilon_o} \tag{14}$$

$$\frac{\partial p_{x1}}{\partial \varepsilon_o}\bigg|_c = \frac{\partial p_{x1}}{\partial x'_1}\frac{\partial x'_1}{\partial x'_o}\frac{\partial x'_o}{\partial \varepsilon_o} + \frac{\partial p_{x1}}{\partial x'_1}\frac{\partial x'_1}{\partial y'_o}\frac{\partial y'_o}{\partial \varepsilon_o} + \frac{\partial p_{x1}}{\partial x'_1}\frac{\partial x'_1}{\partial \varepsilon_o}$$

$$+ \frac{\partial p_{x1}}{\partial y'_1}\frac{\partial y'_1}{\partial y'_o}\frac{\partial y'_o}{\partial \varepsilon_o} + \frac{\partial p_{x1}}{\partial y'_1}\frac{\partial y'_1}{\partial x'_o}\frac{\partial x'_o}{\partial \varepsilon_o} + \frac{\partial p_{x1}}{\partial \varepsilon_1}$$

The derivatives of the transformations between canonical and transport variables are purely kinematic and can be expressed in terms of transport variables. For an ultrarelativistic beam, they are simpler in form, but the argument is unchanged. We then get

$$\frac{\partial p_x}{\partial x'} = \frac{p}{\sqrt{1+x'^2+y'^2}} - \frac{px'^2}{(1+x'^2+y'^2)^{3/2}} \tag{15}$$

$$\frac{\partial p_x}{\partial y'} = -\frac{px'y'}{(1+x'^2+y'^2)^{3/2}}$$

$$\frac{\partial x'}{\partial \varepsilon} = -\frac{x'}{p}(1+x'^2+y'^2)$$

$$\frac{\partial y'}{\partial \varepsilon} = -\frac{y'}{p}(1+x'^2+y'^2)$$

Achromaticity vs. Isochronicity

Incorporation of equations (15) into equations (14) produces quite a mess. However, if we consider only the implications as they apply to each order, some simple conclusions can be derived. To first order, we have

$$\frac{\partial x_1}{\partial \varepsilon_o}\bigg|_c = \frac{\partial x_1}{\partial \varepsilon_o} \tag{16}$$

$$\frac{\partial p_{x1}}{\partial \varepsilon_o}\bigg|_c = p\,\frac{\partial x'_1}{\partial \varepsilon_o}$$

Isochronicity and achromaticity become equivalent with no restrictions. Moving to higher orders, other terms begin to appear and more restrictions need to be imposed. At second order, we need to have a first-order focus in both transverse planes, and the planes need to be independent. In fourth order, we must have unity magnification in both planes.

The partial derivatives of the final transport variables with respect to the initial transport variables are not necessarily the same as the first-order transfer matrix elements. The transfer matrix elements are the partial derivatives evaluated on the reference trajectory. When considering higher orders of achromaticity and isochronicity, the restrictions must also be imposed to an appropriately high order. For example, at third order, the focus must be good to second order. There must be no second-order geometric aberrations.

Now we return to the original theorem. We prove it by induction. Since it is established in first order, the first part of our proof is done. Assume that a beam line is isochronous up to order n-1. Then the n'th order of the vector T_c will be given in terms of the nth order of L and the first-order terms of the matrix product MN. If the nth order terms of L vanish, those of the same order of T_c will also. We define a vector T to be the derivative of the transverse transport variables with respect to ε. Subject to the restrictions described above, then T will vanish to nth order, and the beam line will be achromatic. By the same reasoning, and subject to the same restrictions, the vanishing to a given order of T will imply the vanishing to the same order of L.

The longitudinal higher-order matrix elements covered by this theorem include all except those which are purely energy dependent. Thus terms of the form $(\tau|\varepsilon^n)$, which may also depend on the mass of the particle, are not included. Similarly, the transverse matrix elements are only those which have some momentum dependence, plus those necessary for the stated conditions. Clearly, if all transverse terms of a given order can be made to vanish, then all the terms of L to that order will vanish also.

An example of this theorem to second order has been given by Brown[2]. He has devised a system where all second-order transverse matrix elements can be made to vanish simultaneously. The longitudinal second-order terms then also vanish, except for $(\tau|\varepsilon^2)$.

References

[1] Dragt, Alex, AIP Conference Proceedings No. 87 (1982).
[2] Brown, Karl L., SLAC-PUB-2257 (1979).