

STOCHASTIC COOLING: Recent Theoretical Directions\*

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Summary

A kinetic equation derivation of the stochastic cooling Fokker-Planck equation is discussed. The notion of correlation is introduced to describe both the Schottky spectrum and signal suppression. Generalizations to nonlinear gain and coupling between degrees of freedom are presented. Analysis of bunch beam cooling is included.

Introduction

From its conception by S. van der Meer, little more than a decade ago, the stochastic cooling technique has matured into a powerful tool which has given elementary particle physics the highest available center of mass energies in storage rings. During this brief period, stochastic cooling has been the subject of much experimental and theoretical work, which is too extensive to be chronicled here. The reader is directed to several review papers<sup>1-3</sup> for a thorough account of its development.

The theoretical analysis of stochastic cooling can be approached from a variety of perspectives with a fundamental dichotomy between the frequency domain (spectrum, bandwidth, filters) and the time domain (Fokker-Planck equations). At the heart of the matter is the existence of two distinct time scales: One corresponds to the single particle cooling rate with characteristic times of the order of seconds. The other is that of coherent effects - signal suppression and instabilities - with times typically of the order of milliseconds. This "two-timing" of the physics is basic to an understanding of stochastic cooling and is implicit in most of the literature on the subject.

Another concept which is of importance is that of correlation. It first manifests itself in the frequency variation of the Schottky power spectrum of a particle beam - this being just the frequency domain statement of signal correlation in the time domain. A further effect of correlation is the Schottky signal suppression or shielding of random particle fields observed when a stochastic cooling feedback system is turned on.

In this note a kinetic equation formulation of stochastic cooling is used to clarify the interrelations among the various issues which have been highlighted. Recent developments which address coupling between degrees of freedom, gain nonlinearities, and the cooling of bunched beams will be reviewed. Signal suppression and enhancement from feedback systems and general machine impedances will also be discussed.

Single Particle Interaction

Stochastic cooling is the damping of betatron oscillations and momentum spread of a particle beam by a feedback system. In its simplest form, a pickup electrode detects the transverse positions or momenta

of particles in a storage ring, and the signal produced is amplified and applied downstream to a kicker. The time delay of the cable and electronics is designed to match the transit time of the particles along the arc of the storage ring between the pickup and kicker so that an individual particle receives the amplified version of the signal it produced at the pickup. If there were only a single particle in the ring, it is obvious that betatron oscillations and momentum offset could be damped. The feedback system produces an inherently dissipative self-interaction which derives from a large scale asymmetry between the pickup and kicker fields introduced by the high gain amplifier chain.

Consider a single particle circulating in a ring at angular revolution frequency  $\omega = 2\pi/T$ . The current at a longitudinal pickup can be described as a series of delta functions

$$I(t) = e \sum_{n=-\infty}^{+\infty} \delta(t - t_0 - nT) = \frac{e\omega}{2\pi} \sum_n e^{in\omega(t-t_0)} \quad (1)$$

where  $t_0$  is some arbitrary time at which the particle is in the pickup. The signal produced at this pickup can be amplified and applied to a kicker. The signal at the kicker from all beam particles will be of the form

$$k(t) = \frac{2\pi}{\omega_0} \sum_j \sum_n G(n\omega_j, x_j) e^{in\omega_j(t-t_{0j})} \quad (2)$$

where  $G(\Omega, x_j)$  represents the electronic transfer character of the system at electronic frequency  $\Omega$  and particle energy  $x_j$ . Let us now focus attention on a single particle, say the  $i$ th. It does not experience the signal at the kicker continuously, but rather samples it once every revolution period. The correction signal it receives is

$$k_i(t) = \sum_m \delta(t - t_i - mT_i) k(t) \quad (3)$$

or expanding the delta function

$$k_i(t) = \sum_j \sum_n \sum_m G(n\omega_j, x_j) e^{i(n\omega_j - m\omega_j)t} e^{-i(n\omega_j t_{0j} - m\omega_j t_{i1})} \quad (4)$$

Since cooling is a slow process relative to the revolution frequency, the effect of the rapidly oscillating part of  $k_i$  is negligible unless there is a rapidly growing instability. The slowly varying part of the self-interaction term is

$$k_i \Big|_{\text{self}} = \sum_n G(n\omega_i, x_i) e^{-in\omega_i(t_{0i} - t_{i1})} \quad (5)$$

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With a judicious choice of gain through filters and pickup design a dissipative, velocity dependent interaction is obtained. Because of the large gain, effects of pickup fields can be neglected. Similar sampling arguments can be applied to arbitrary particle orbits; for example, synchrotron oscillations. For stochastic cooling it is this direct single particle self-interaction which increases the phase space density. This is in contrast to electron cooling, where the relaxation process is driven by polarization effects. In stochastic cooling polarization phenomena are important, but describe the interference of neighboring particles.

#### Kinetic Equations for Non-Liouvillian Systems

Particles receive successive kicks from each other which have long time coherence if the particles are near both in azimuth and velocity. This coherence leads to correlation of the particles' phase space coordinates. In a plasma physics context such correlation is expressed in terms of Debye screening of single particle random fields, the analogue of Schottky signal suppression. For plasmas the most self-consistent description of this shielding is given by the Lenard-Balescu equation, which requires the introduction of the two particle joint distribution in addition to the usual one particle distribution of Vlasov theory. It is this framework which is basic to a description of electron cooling. For stochastic cooling a similar set of equations can be derived, with modifications coming from the self-interaction term.

For simplicity, we will first discuss one dimensional longitudinal cooling. (The essentials of the argument are unchanged if the variables are interpreted as vectors.) The stage for our analysis is a  $2N$ -dimensional ensemble space whose elements are vectors  $(q_1, p_1, \dots, q_N, p_N)$ . Each vector represents a whole system of  $N$  particles with positions  $q_i$  and momentum  $p_i$ . Consider an ensemble distribution  $D(q_1, p_1, \dots, q_N, p_N)$  describing a collection of these  $N$  particles. Conservation of the number of ensemble systems is expressed by

$$\frac{\partial D}{\partial t} + \vec{v} \cdot (\vec{D}) = 0 \quad (6)$$

where  $\vec{v} = (\dot{q}_1, \dot{p}_1, \dots, \dot{q}_N, \dot{p}_N)$ . As we have seen, the single particle dynamics are of the form

$$\dot{p}_i = \sum_j G(q_i, q_j, p_j) \quad \dot{q}_i = Q(p_i) \quad (7)$$

Define the one and two particle distributions  $f_1$  and  $f_2$  by integrating  $D$  over  $2N-2$  and  $2N-4$  variables, respectively. Define the two particle correlation by

$$g(q_1, p_1, q_2, p_2, t) = f_2(q_1, p_1, q_2, p_2, t) - f_1(q_1, p_1, t) f_1(q_2, p_2, t) \quad (8)$$

Integration of equation (6) over  $2N-2$  variables yields the stochastic cooling analogue of the Vlasov/Boltzmann equation:

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \dot{q}_1 \frac{\partial f_1}{\partial q_1} + (N) \frac{\partial f_1}{\partial p_1} \int dq_2 dp_2 G(q_1, q_2, p_2) f_1(q_2, p_2, t) \\ = - \frac{\partial}{\partial p_1} \left[ G(q_1, q_1, p_1) f_1(q_1, p_1, t) \right] \\ - (N) \frac{\partial}{\partial p_1} \int dq_2 dp_2 G(q_1, q_2, p_2) \\ g(q_1, p_1, q_2, p_2, t) \end{aligned} \quad (9)$$

With the RHS set to zero we have the usual Vlasov equation. The first term on the RHS describes the direct self-interaction which increases phase space density. The second term on the RHS contains collision effects (Schottky noise heating) which may be shielded.

The two particle correlation equation is obtained by integrating equation (6) over  $2N-4$  variables, yielding

$$\begin{aligned} \frac{\partial g}{\partial t} + \dot{q}_1 \frac{\partial g}{\partial q_1} + \dot{q}_2 \frac{\partial g}{\partial q_2} \\ + N \frac{\partial g}{\partial p_1} \int dq_3 dp_3 G(q_1, q_3, p_3) f_1(q_3, p_3, t) \\ + N \frac{\partial g}{\partial p_2} \int dq_3 dp_3 G(q_2, q_3, p_3) f_1(q_3, p_3, t) = \\ - \frac{\partial}{\partial p_1} \left[ G(q_1, q_2, p_2) f_1(q_1, p_1, t) f_1(q_2, p_2, t) \right] \\ - \frac{\partial}{\partial p_2} \left[ G(q_2, q_1, p_1) f_1(q_1, p_1, t) f_1(q_2, p_2, t) \right] \\ - N \frac{\partial f_1}{\partial p_1} \int dq_3 dp_3 G(q_1, q_3, p_3) g(q_2, p_2, q_3, p_3, t) \\ - N \frac{\partial f_1}{\partial p_2} \int dq_3 dp_3 G(q_2, q_3, p_3) g(q_3, p_3, q_1, p_1, t) \end{aligned} \quad (10)$$

As in the Lenard-Balescu equation, terms of the order of 3-particle correlations have been dropped. In the plasma context, the justification is the existence of a small parameter (the ratio of interaction energy to thermal energy) which orders the correlations. For stochastic cooling, as will be manifest in the signal suppression expression, this small parameter is the ratio of the damping rate to the revolution frequency spread.

In equation (10), the first two terms on the RHS describe the direct effects of beam particles perturbing each other, including both polarization (feedback through the beam) and collision (Schottky heating). The last two terms describe how existing coherence limits correlation growth. The  $\dot{q}$  terms on the LHS effect mixing through frequency spread and enhance the interaction of particles neighboring in frequency.

Equation (10) can be solved under the assumption that the relaxation time of the correlation  $g$  is fast on the scale of variation of  $f_1$ . Physically,  $g$  describes the buildup of coherence; e.g., single suppression or instability, which have rise times of a few milliseconds. The changes in  $f_1$ , on the other hand, have a time scale of seconds.

For simplicity let us take as our variables azimuthal angle  $\theta$  and energy error  $x = (E - E_0)$ . Expand the gain

$$G(\theta_i, \theta_j, x_j) = \sum_n G_n(x_j) e^{in(\theta_i - \theta_j)} \quad (11)$$

and

$$g(\theta_1, \theta_2, x_1, x_2, t) = \sum_\ell g_\ell(x_1, x_2, t) e^{i\ell(\theta_1 - \theta_2)} \quad (12)$$

Define

$$H_\ell(x_1) = N \int dx_2 G_{-\ell}(x_2) g_\ell(x_1, x_2) + f(x_1) G_{-\ell}(x_1) \quad (13)$$

For a uniform beam, equation (9) yields

$$\frac{\partial f}{\partial t}(x, t) = - \frac{\partial}{\partial x} \left[ \sum_\ell H_\ell(x) \right] \quad (14)$$

After Laplace transforming with  $f$  considered constant, we have from equation (10) the integral equation

$$H_{\pm|\ell|}(x_1) = G_{\mp|\ell|}(x_1) f(x_1) / \epsilon_{\pm|\ell|} \\ - \frac{1}{\epsilon_{\pm|\ell|}} \frac{N}{|\ell|} \frac{\partial f_1}{\partial x_1} \int_{n \geq 0_+} dx_2 \frac{H_{\mp\ell}(x_2)}{n \pm i(\omega_1 - \omega_2)} G_{\mp|\ell|}(x_2) \quad (15)$$

where

$$\epsilon_{\pm|\ell|}(x_1) = 1 + \frac{N}{|\ell|} \int_{n \geq 0_+} dx_2 \frac{\frac{\partial f}{\partial x_2} G_{\mp|\ell|}(x_2)}{n \pm i(\omega_1 - \omega_2)} \quad (16)$$

Except for the details of the gain  $G$ , this is the integral equation of Lenard-Balescu. The solution requires some complex plane gymnastics, too complicated to be described here. However, an iterative solution, assuming the second term on the RHS is small, yields on insertion into (14) the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = - \sum_\ell \left\{ \frac{\partial}{\partial x} \left[ \frac{G_\ell(x) f(x, t)}{\epsilon_{-\ell}(x)} \right] - \frac{\partial}{\partial x} \left[ \frac{N}{|\ell|} \left| \frac{\partial x}{\partial \omega} \right| \left| \frac{G_\ell}{\epsilon_{-\ell}} \right|^2 \frac{\partial f}{\partial x} f \right] \right\} \quad (17)$$

+ (principle value integrals)

The exact solution yields the same result, without the principle value integrals.

### Multidimensional Systems

The above analysis can be generalized to all three degrees of freedom to include coupling, nonlinearity, and betatron oscillations.

We take as our canonical coordinates three dimensional action angle variables,  $\underline{I}$  and  $\underline{\psi}$ . The gain will now be a vector object

$$\underline{G}(\underline{I}_1, \underline{\psi}_1; \underline{I}_2, \underline{\psi}_2) = \sum_{\underline{n}_1} \sum_{\underline{n}_2} \underline{G}_{\underline{n}_1 \underline{n}_2}(\underline{I}_1, \underline{I}_2) e^{i \underline{n}_1 \cdot \underline{\psi}_1 + i \underline{n}_2 \cdot \underline{\psi}_2} \quad (18)$$

which allows in particular for nonlinear coupling. The gain  $\underline{G}$  is defined through the relation

$$\dot{\underline{I}}_i = \sum_j \underline{G}(\underline{I}_i, \underline{\psi}_i; \underline{I}_j, \underline{\psi}_j) \quad (19)$$

obtained by averaging the sampled signal. The one particle distribution equation becomes<sup>4</sup>

$$\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial \underline{I}} \cdot \sum_n \underline{G}_{n-n}(\underline{I}, \underline{I}) f_0(\underline{I}, t) \\ = - \frac{\partial}{\partial \underline{I}} \cdot \left[ \sum_n \underline{R}_{nn}^*(\underline{I}, \underline{I}) \right] \quad (20)$$

where

$$\underline{R}_{\underline{n}_1 \underline{n}_2}(\underline{I}_1 \underline{I}_2) = N \sum_{\underline{n}_3} \int d\underline{I}_3 \underline{G}_{\underline{n}_1 \underline{n}_3}^*(\underline{I}_1 \underline{I}_3) \cdot \underline{g}_{\underline{n}_2 \underline{n}_3}(\underline{I}_2 \underline{I}_3 t) \quad (21)$$

and  $g$  satisfies the vector analogue of (10). Without signal suppression, the Fokker-Planck equation can be written in the form

$$\frac{\partial f_0}{\partial t} = - \frac{\partial}{\partial \underline{I}} \cdot [\underline{F}(\underline{I}) f(\underline{I})] \\ + \frac{1}{2} \frac{\partial}{\partial \underline{I}} \cdot \underline{D}(\underline{I}) \cdot \frac{\partial f_0}{\partial \underline{I}} \quad (22)$$

where

$$\underline{F}(\underline{I}) = \sum_n \underline{G}_{n-n}(\underline{I}, \underline{I})$$

$$\underline{D}(\underline{I}) = (2\pi) N \sum_n \sum_{n'} \cdot$$

$$\int d\underline{I}' \delta(\underline{n}' \cdot \underline{\omega}' - \underline{n} \cdot \underline{\omega}) \underline{G}_{\underline{n} \underline{n}'}(\underline{I}, \underline{I}') \underline{G}_{\underline{n} \underline{n}'}^*(\underline{I}, \underline{I}') \cdot f_0(\underline{I}') \quad (23)$$

Note the  $\underline{n}' \cdot \underline{\omega}' - \underline{n} \cdot \underline{\omega}$  argument of the delta function, which couples particles of different frequencies. This band overlap effect also appears

in the  $\epsilon$  suppression factors and will be discussed later.

A few comments are now in order. The  $\epsilon$  factor is described by the same dispersion integrals that a Vlasov equation calculation would yield for coherent response. Analogously, the Debye shielding of a plasma can be derived from a test particle approach and then used to modify the Boltzmann equation with collisions. Secondly, for the most general gain function, either the Vlasov or correlation equation is infinite dimensional and contains all the pitfalls that have frustrated single bunch instability analysis. Finally, the correlation equation (10) does not contain the self interaction term, and can be applied to any machine impedance. The related  $\epsilon$ -factor describes the deformation of Schottky signals due to correlations developing between the beam particles. The Schottky spectrum of the  $q^{\text{th}}$  line is modified by

$$f(\omega) \rightarrow \frac{f(\omega)}{|\epsilon_q|^2} \quad (24)$$

Similarly, the tensor  $D$  which describes the multi-dimensional Schottky noise will be modified by an  $\epsilon$  tensor with

$$\underline{D} \rightarrow \underline{\epsilon}^{-1} \cdot \underline{D} \cdot \underline{\epsilon}^{-1+}$$

This  $\epsilon$  tensor is most easily described through the associated Vlasov equation. Amplifier noise can be introduced through modification of  $\underline{D}$ .

#### Bunch Beam Cooling

In particular, the general formulation can be applied to bunch beam cooling. For longitudinal cooling the gain  $G$  is of the form

$$G_{\mu\nu}(J^i J^j) = \frac{1}{\omega_s(J^i)} \sum_{m=-\infty}^{+\infty} \left( \frac{\nu}{-m} \right) G(m\omega_0 + \mu\omega_s)(J^j) \\ \times J_{\mu}(m\sqrt{2J^i}) J_{\nu}(-m\sqrt{2J^j})$$

where the Bessel functions  $J$  describe the synchrotron orbits with sampling. When  $G_{\mu\nu}$  is used to evaluate  $\underline{D}$ , interference terms will arise between the  $m\omega_0$  harmonics. This feature is a consequence of the nonstationary nature of bunch beam Schottky noise, which is not time translation invariant.

Some approximation is necessary to solve the infinite dimensional suppression equation. For example, keeping only the dominant pole terms yields

$$\epsilon^{\mu}(J) = 1 + \pi N \frac{\partial f_0}{\partial \omega_s} \sum_n \frac{J_{\nu}^2(n\sqrt{2J})}{n} G(n\omega_0 + \mu\omega_s)$$

and indicates the importance of synchrotron frequency spread for effective cooling. Note that the gain for all harmonics  $n$  enter into the  $\epsilon^{\mu}$  suppression, weighted by  $J_{\nu}(n\sqrt{2J})$ . This interference effectively introduces the local particle density on the bunch in determining the optimum cooling rates. Numerical studies indicate that with synchrotron frequency spread sufficient for satellite band overlap and filtering of the coherent bunch signal, it is the high

local particle densities, not bad mixing, which limits bunch beam cooling rates.

#### Schottky Band Overlap and Signal Suppression

From equation (23) for  $D$ , it is apparent that particles with different revolution frequencies  $\omega, \omega'$  can interact if  $n\omega = m\omega'$ . For  $n \neq m$ , this is the condition of Schottky band overlap. From a Vlasov approach which includes the local character of the feedback system, it can be shown that the  $\epsilon$  factor has a similar property.<sup>5,6</sup>

Consider, for example, the longitudinal Vlasov equation

$$\frac{\partial f}{\partial t} + \omega \frac{\partial f}{\partial \theta} + \frac{F(\theta, t)}{\omega} \frac{\partial}{\partial x} (\omega f_0) = 0 \quad (25)$$

The localized interaction  $F$  for a feedback system is of the form

$$F(\theta, t) = -2\pi \delta(\theta - \theta_k) N \int dx' \omega' \hat{G}(x', t - t') f(x', \theta_p, t)$$

where  $\theta_p$  and  $\theta_k$  are the pickup and kicker azimuth, respectively. Note the  $\delta$ -function character of the kick, and that only the value of  $f$  at the pickup produces signal. The associated  $\epsilon$  is found to be of the form

$$\epsilon(m\omega) = 1 + \sum_l \int dx' N G(x', m\omega) \frac{\partial}{\partial x'} (\omega' f) \frac{e^{i l(\theta_k - \theta_p)}}{i(m\omega - l\omega')}$$

where  $G$  is the Fourier transform of  $\hat{G}$ . The coupling of the overlapping bands is now manifest. The sum can be expressed in closed form and is causal and localized spatially. Because of this locality,  $\epsilon$  exhibits large gain instabilities corresponding to feedback overdamping, which would disappear in a smooth approximation. In addition, if there is signal suppression within a Schottky band,  $\epsilon(\Omega)$  requires signal enhancement outside the band.

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