

THE INSTABILITY OF AN ANNULAR ELECTRON BEAM PROPAGATING ALONG A GUIDED MAGNETIC FIELD

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SUMMARY

The instabilities of an infinitely long and azimuthally symmetric annular electron beam propagating along an external applied magnetic field are investigated using the fluid-Maxwell theory. Unlike the treatment in the conventional diocotron instability, full electromagnetic perturbations have been emphasized and no approximation has been made such as $k_{\parallel} \sim 0$ and $\omega \sim ck\beta$ in the macroscopic cold fluid description. There have been found two distinct and different instabilities depending on the value of doppler-shifted eigenfrequency $\omega - ck\beta$. The diocotron instability dominates in the low frequency region while cyclotron resonance perturbation plays a major role in the high frequency domain. In the case of a sharp boundary density profile, the growth rates versus different wavenumber k are shown for both instabilities in the beam parameters of our interest.

I. INTRODUCTION

The diocotron instability of a hollow electron beam has been known for a long time since the early crossed field microwave magnetron. For an annular electron beam in the cylindrical geometry, Buneman¹ has considered the diocotron instability in the regime of $\omega_{pb} = \omega_c$ while Levy² has examined the instability in the case of low beam density $\omega_{pb} < \omega_c$, where ω_{pb} and ω_c stand for the electron plasma and cyclotron frequency respectively. Since then numerous theoretical work have been done and most of them consider only perturbations with sufficiently long wavelength. Experimentally³, there are some evidence that the lower order diocotron modes have been found in the damaged plate after the target interaction of the annular electron beam. Recently⁴, the filamentation instability ($\ell > 2$) of an annular electron beam along a uniform magnetic field has been studied by using the Vlasov-Maxwell equations. Here, a more complete full electromagnetic treatment of the fluid-Maxwell theory is conducted without any approximations which have been made previously. We want to consider the perturbations not only in azimuthal ($\ell \neq 0$) but also in axial direction ($k_z \neq 0$). In addition, the model developed here will apply to all range of beam density and other beam parameters except the requirement of a thin hollow beam. Most importantly, we want to examine the validity of the previous $\omega - ck\beta$ assumption in the treatment of the conventional diocotron instability.

II. EQUILIBRIUM AND ASSUMPTION

The equilibrium configuration consists of a cylindrically symmetric annular electron beam that is infinite in the axial direction and aligned parallel to a uniform applied magnetic field. The radial thickness of the hollowing beam denoted by $2a$ is assumed small in comparison with the mean equilibrium radius R_0 . The beam under consideration is characterized by the charge q , mass m , axial velocity $c\beta$ and density profile n respectively. The flow of electrons can be considered laminar provided that the beam current is much smaller than the Alfvén-Lawson space-charge limiting current⁵

$$I < 17000 \beta \gamma$$

Furthermore, the beam electron motion is taken to be paraxial so that the axial velocity is very large compared to the transverse velocity and is considered to be a constant. We introduce a cylindrical polar coordinate system (r, θ, z) with the z axis coinciding with the axis of symmetry. Analysis of beam dynamic properties is based on a macroscopic cold fluid model. The equation of electron and momentum conservation for the electron fluid can be expressed in the relativistic form as

$$\partial n / \partial t + \nabla \cdot (n \underline{V}) = 0 \quad (1)$$

$$(\partial / \partial t + \underline{V} \cdot \nabla) \underline{V} = \frac{q}{\gamma m} (\underline{E} + \underline{V} \times \underline{B} / c) \quad (2)$$

where $n(\underline{x}, t)$ and $\underline{V}(\underline{x}, t)$ are the density and mean velocity of an electron fluid element, $\underline{E}(\underline{x}, t)$ and $\underline{B}(\underline{x}, t)$ are the electric and magnetic fields respectively. γ and β are the standard relativistic quantities and c is the speed of light in vacuum. The self-induced electric and magnetic field can be related to the beam density and current by the Maxwell's equations.

In the steady state ($\partial / \partial t = 0$) the beam is assumed azimuthally symmetric ($\partial / \partial \theta = 0$) and infinite long and uniform in the axial direction ($\partial / \partial z = 0$). The equilibrium force balance due to electric and magnetic field in the radial direction gives the angular velocity $\omega_b(r)$ of an electron fluid element in slow rotational equilibrium.

$$\omega_b(r) = [1 - (R_0 - a)^2 / r^2] \omega_{pb}^2 / (2\gamma^2 \omega_c^2) \quad (3)$$

where $\omega_{pb}^2 = 4\pi n q^2 / \gamma m$ and $\omega_c = q B_0 / c \gamma m$ are the electron plasma frequency square and cyclotron frequency respectively, and B_0 is the guided magnetic field. For a thin beam, although $\omega_b(r)$ is weakly dependent on r but introduces the diocotron instability

III. STABILITY ANALYSIS

We assume, without loss of generality, that all the perturbed quantities have the following wave form with the sinusoidal time dependence and spatial variation

$$\delta \phi(\underline{x}, t) = \delta \phi(r) \exp [i(\ell \theta + k z - \omega t)] \quad (4)$$

where the oscillating angular frequency ω is assumed to be complex with $\text{Im}(\omega) > 0$, ℓ is the azimuthal harmonic number and k is the propagation wavenumber in the axial direction. We use the linearized fluid-Maxwell equations to investigate the general electromagnetic perturbations for $\ell \geq 1$ and any arbitrary wavenumber k . Let us choose the transverse magnetic (TM) modes such that the magnetic field lies in the cross-sectional plane. Since all the transverse fields for the TM mode can be expressed in terms of the z -component of the electric field, the determination of the transverse fields therefore can be expressed in terms of δE_z according to

$$\delta B_t = (\omega^2 / c^2 - k^2)^{-1} i \frac{\omega}{c} \hat{e}_z \times \nabla_t \delta E_z$$

$$\delta E_t = (\omega^2 / c^2 - k^2)^{-1} \nabla_t (\partial \delta E_z / \partial z)$$

From the perturbed Maxwell equation, it is straightforward to express, after some algebraic manipulations, that the relationship between the perturbed field δE_z and the source terms δn and δJ_z

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \delta E_z = 4\pi ik \left(q \delta n - \frac{\omega}{kc} \delta J_z \right). \quad (5)$$

where δn and δJ_z can be obtained by solving a set of first order perturbation equation from equations (1) and (2) and expressed in terms of δE_z . Therefore, equation (5) together with boundary conditions on the field component represented by the scalar function δE_z , specifies a two-point boundary eigenvalue problem which can be solved numerically. For the special case of a square density profile for the hollow beam, the above mentioned procedure, with some straightforward but tedious algebraic manipulations, rewrites equation (5) in the final form as

$$\left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[r(1-S_1) \frac{\partial}{\partial r} \right] - \frac{\ell^2}{r^2} (1-S_1) - \left(k^2 + \frac{\omega^2}{c^2} \right) \right\} \delta E_z(r) = \frac{\ell}{r} S_2 [\delta(r-R_0+a) - \delta(r-R_0-a)] \delta E_z(r) + \frac{1}{rR_0} S_3 \delta E_z(R_0) \quad (6)$$

with substitutions

$$S_1 \equiv \frac{\omega_{pb}^2}{\Omega_0^2 - \omega_c^2} \frac{(1 - \frac{\omega\beta}{ck})^2}{1 - (\frac{\omega}{ck})^2} \quad (7)$$

$$S_2 \equiv - \frac{\omega_{pb}^2 \omega_c}{(\Omega^2 - \omega_c^2) \Omega} \frac{(1 - \frac{\omega\beta}{ck})^2}{1 - (\frac{\omega}{ck})^2} \quad (8)$$

$$S_3 \equiv - \left[\frac{(ck - \omega\beta)ck}{\Omega_0^2} + \frac{\omega}{\Omega_0} \right] \left(\frac{R_0 \omega_{pb}}{c} \right)^2 \left[1 + \frac{\ell\omega\omega_b}{(ck)^2 - \omega^2} \right] \quad (9)$$

where $\Omega \equiv \omega - \ell\omega_b(r) - ck\beta$ and $\Omega_0 \equiv \omega - \ell\omega_b(R_0) - ck\beta$. The contribution to the right hand side of equation (6) becomes two delta function at the sharp boundaries and one additional term which is assumed inversely proportional to r only. The rest of the term is evaluated approximately at R_0 for a thin annular beam; the self-induced magnetic field B_0 has been neglected in deriving equation (6) and $\omega_b \ll \omega_c$ has been used. The piece-wise solutions for the homogeneous equation of equation (6) can be expressed as the eigenfunction is continuous at each boundary and vanish at $r = 0$ and $r = \infty$, where the conducting wall has been removed so that $R_c = \infty$. The effect of the delta function can be considered by multiplying both sides of r and integrating over the infinitesimal interval from $r(1-\epsilon)$ to $r(1+\epsilon)$ with $\epsilon \rightarrow 0$ in the vicinity of $r=R_1$ and R_2 respectively. The dispersion relation is obtained by following the steps from Eq. (88) to Eq. (99) of Ref.6, the final results can be expressed by a 2x2 matrix as

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = 0 \quad (10)$$

for $\ell = 1$

$$A_{11} = S_4 - [S_1 + S_2(R_1)] R_1 / R_0$$

$$A_{12} = S_4 - [2 - S_1 + S_2(R_1)] R_0 / R_1$$

$$A_{21} = S_5 - [2 - S_1 - S_2(R_2)] R_2 / R_0 \quad (11)$$

$$A_{22} = S_5 - [S_1 - S_2(R_2)] R_0 / R_2$$

with the substitutions

$$S_4 \equiv \frac{S_3}{2} \frac{R_1}{R_0} - [S_1 + S_2(R_1)] \frac{1}{2} \frac{R_1}{R_0} \ln \frac{R_1}{R_0} \frac{S_3}{1 - S_1}$$

$$S_5 \equiv - \frac{S_3}{2} \frac{R_2}{R_0} - [2 - S_1 - S_2(R_2)] \frac{1}{2} \frac{R_2}{R_0} \ln \frac{R_2}{R_0} \frac{S_3}{1 - S_1}.$$

It is important to note that S_1 and S_3 from equations (7) and (9) are independent of r . However, S_2 of equation (8) is a function of r through the variation of $\omega_b(r)$. Similarly, for $\ell \geq 2$

$$A_{11} = R_1^{\ell-1} S_2(R_1) - R_0^{\ell-1} S_6$$

$$A_{12} = R_1^{-\ell-1} [2 - 2S_1 + S_2(R_1)] - R_0^{-\ell-1} S_6 \quad (12)$$

$$A_{21} = R_2^{\ell-1} [2 - 2S_1 - S_2(R_2)] + R_0^{\ell-1} S_7$$

$$A_{22} = -R_2^{-\ell-1} S_2(R_2) + R_0^{-\ell-1} S_7$$

where the substitutions have been used

$$S_6 = \frac{S_3 [S_2(R_1) + (1 - S_1)(\ell - 1) / \ell]}{(\ell^2 - 1)(1 - S_1) + S_3}$$

$$S_7 = \frac{S_3 [S_2(R_2) - (1 - S_1)(\ell + 1) / \ell]}{(\ell^2 - 1)(1 - S_1) + S_3}$$

IV. NUMERICAL RESULTS AND CONCLUSIONS

(a) Diocotron instability

The growth rate and Doppler-shifted real frequency, both have been normalized to the beam plasma frequency, are plotted versus normalized axial wavenumber in Fig. 1a and 1b respectively for the azimuthal

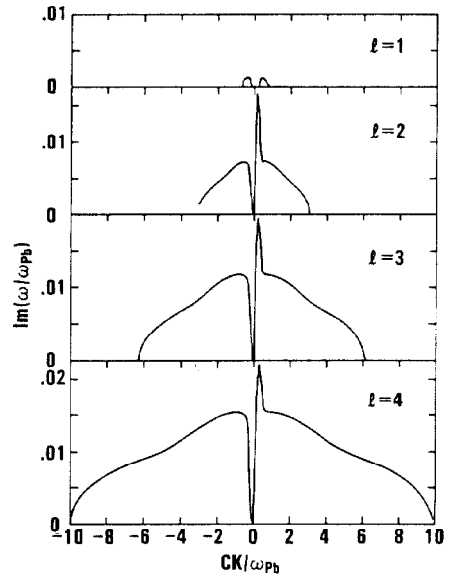


Fig. 1a. The diocotron growth rate versus wavenumber for different azimuthal number. The beam parameters are $\gamma = 2$, $a/R_0 = .05$, $\omega_{pb} R_0 / c = .05$, $\omega_{pb} / \omega_c = .5$

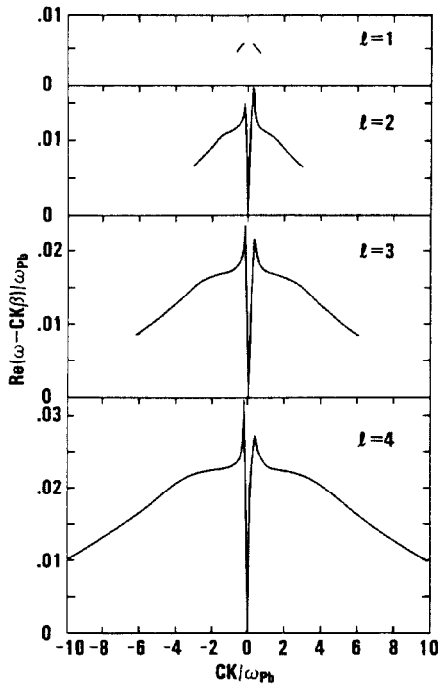


Fig. 1b. The real doppler-shift real frequency of the diocotron instability corresponding to Fig. 1a

mode $l=1$ and a few higher modes. Note that it is always stable in the limit of $k=0$. $\text{Re}(\omega - ck\beta)$ remains very small which characterizes the low frequency diocotron perturbations. The second term on the RHS of Eq. (6) becomes very important especially for large k . If $\omega = ck\beta$ is applied to Eq. (6), then Eq. (6) becomes⁷ Eq. (2.6.20) of Davison of the conventional diocotron instability. The symmetry, with respect to k , can be seen except those near $k=0$. As the value of $R_0\omega_{pb}/c$ increases, Eq. (9) involving the non-symmetric term with respect to k becomes important. The results obtained agree qualitatively to those using the Vlasov system of equations to obtain a kinetic description of the situation.

(b) Cyclotron resonance instability

It is clear that the diocotron instability occurs whenever $\omega \sim ck\beta$ or $\Omega \ll \omega_c$ as demonstrated in (a). Nonetheless, in the high Ω regime (i.e., $\Omega \gg \omega_c$), there is a distinct and different instability called cyclotron resonance oscillation. The dispersion relation of Eq. (10) is solved numerically for the growth rate and Doppler-shifted real frequency which are illustrated in Fig. 2. Unlike the diocotron instability, the growth rate and Doppler-shifted real frequency are very large. The instability is almost independent of l , γ , a/R_0 and $\omega_{pb}R_0/c$. Since Ω is very large, the second term on the RHS of Eq. (6) can be neglected in the calculation. In order to keep $|\Omega| \sim |\omega_c|$, $\text{Re}(\omega)$ and $ck\beta$ could have almost the same magnitude but different signs, (i.e., $\omega \sim -ck\beta$). The cyclotron resonance wave has been observed in the particle simulation and applied to Texas collective ion accelerator experimentally. It has not been seen because of the approximation $\omega \sim ck\beta$ in the treatment of the conventional diocotron instability.

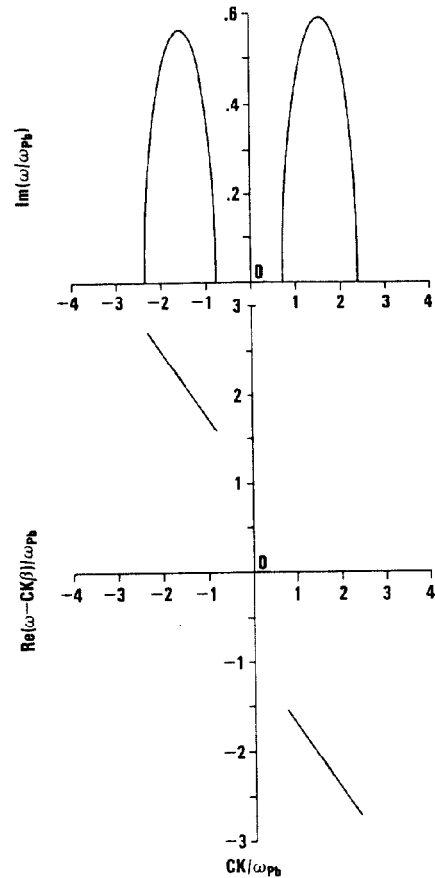


Fig. 2. The growth rate of cyclotron resonance wave versus wavenumber for the beam parameters similar to Fig. 1

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