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LONGITUDINAL STABILITY OF A COASTING BEAM IN A CORRUGATED VACUUM CHAMBER

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Summary

Widenings of a vacuum chamber may function as resonant cavities, which can promote longitudinal instabilities. We have reexamined the stability question in such a situation, following earlier work of Keil and Zotter. We solve the full linearized Vlasov-Maxwell equations for a model vacuum chamber, using an improved computational method which is efficient for virtually any choice of geometrical parameters, in particular for large ring radius R. Because of mode coupling induced by the corrugations, the dispersion relation that determines the growth time of an instability involves an infinite-dimensional impedance matrix, rather than the usual single impedance. We give a method to compute the growth time τ, and find an explicit formula for τ as an oscillating function of the mean revolution frequency Ω, with sharp minima at points where the real part of a cavity resonance frequency is equal to an harmonic of Ω (plus a small shift). An illustration for parameters of interest in ion beam fusion is included.

Equation for cavity mode coefficients with Vlasov self-consistency

As in Ref.1, the vacuum chamber is modeled as a straight pipe of radius b with widenings of radius d and length g; the widenings appear with period $2\pi R$ in the longitudinal variable z. The beam has radius a and is uniform over a cross section. With the assumption of azimuthal symmetry of fields and standard resistive wall boundary conditions, the longitudinal Vlasov equation (linearized about a uniform beam configuration) and the coupled Maxwell equations are subjected to a Laplace transform in time and a Fourier transform in z. The result is an infinite linear system of equations for $\bar{A}_m(p)$, $f_m(p)$, and $\bar{D}_s(p)$, where $p = -i\omega$ is the Laplace transform variable; $\bar{A}_m(p)$ and $f_m(p)$ are coefficients in Fourier developments (period $2\pi R$) of the perturbed electric field and distribution function, respectively, along the axis of the tube, while $\bar{D}(p)$ is the coefficient in a development (period 2g) of the electric field in the cavity region ($r > b$, $-g/2 < z < g/2$). The equations are generalizations of those of Keil and Zotter, Ref.1, and we follow their notation.

The conventional dispersion relation for determination of growth rates is obtained by eliminating f_m and $\bar{D}_s$ to get a set of equations for the $\bar{A}_m$ alone. An infinite-dimensional impedance matrix appears as the coefficient of the plasma dispersion function in those equations. As we shall explain elsewhere, the usual single-impedance description can be retrieved, but it is hard to judge the error in making that reduction. We prefer a different approach in which f_m and $\bar{A}_m$ are eliminated to get an equation for $\bar{D}_s$ alone. Some of the advantages of the $\bar{D}$ equation were recognized by Keil and Zotter; we find further advantages in extending their work. When the cavities are fairly deep (say $b/d < 1/2$), the Fourier modes of the field they contain are not far from being the normal modes of the system. Mathematically, that means that the matrix of the $\bar{D}$ equation is close to being diagonal, and that the off-diagonal matrix elements may be treated by a perturbation method. The $\bar{A}$ equation is nearly diagonal if $b \simeq d$, and should be used in that limit.

To simplify notation in this brief report, we neglect resistivity of the chamber walls, except on the cylindrical surface of the cavity. By using a generalized Fourier series with anharmonic wave numbers, one can treat the cavity end walls as well as the cylindrical surfaces, and derive equations with full account of resistivity which are just as tractable in computations as those stated below. This extension will be reported elsewhere; it is quantitatively important.

The equation for $\bar{D}_s$ with Vlasov self-consistency is

$$\bar{D}_s = R_s \sum_{t=0}^{\infty} (E_{st} + S_{st})\bar{D}_t + R_s D_s^{(o)} , \quad s=0,1,\ldots \tag{1}$$

where E_{st} is the kernel found in Ref.1, purely electromagnetic in origin, and S_{st} expresses the Vlasov self-interaction. The inhomogeneous term $D_s^{(o)}$ arises from the Laplace transform and is a known function of initial values of the fields and distribution function. The function R_s is a certain ratio of Bessel function products, entailing the cavity resistivity; it is equal to R_{ss} of Eq.(1.22), Ref.1. The kernel matrices are

$$E_{st} = (g/2\pi R) \sum_{m=-\infty}^{\infty} N_{sm}^{\dagger} N_{mt} I_1(\chi_m b)/\chi_m b I_o(\chi_m b), \tag{2}$$

$$S_{st} = (g/2\pi R) \sum_{m=-\infty}^{\infty} N_{sm}^{\dagger} N_{mt} I_1(\chi_m b)/\chi_m a I_o^2(\chi_m b) \frac{1}{W_m^{-1} + \Theta_m} \tag{3}$$

where $N_{mt} = N_{tm}^{\dagger *}$ is a geometrical factor, $|N_{mt}|^2$ being the transit-time factor. With $k_m = m/R$,

$$N_{mt} = \frac{2k_m g}{(k_m g)^2 - (t\pi)^2} \begin{cases} \sin(k_m g/2) , & t \text{ even} \\ -i \cos(k_m g/2) , & t \text{ odd} \end{cases} \tag{4}$$

Further, $\chi_m^2 = k_m^2 - (\omega/c)^2$ and Θ_m represents a reactive effect associated with the tube,

$$\Theta_m = -\chi_m a \left(K_1(\chi_m a) + I_1(\chi_m a) K_o(\chi_m b)/I_o(\chi_m b)\right) + 1. \tag{5}$$

Finally, W_m is the plasma dispersion function,

$$W_m = (\omega_p^2/k_m v) \int_{-\infty}^{\infty} dv\, f_o'(v) \left(\omega - k_m v\right)^{-1} , \tag{6}$$

$\omega_p^2 = n(Ze)^2/\varepsilon_o M$ being the squared plasma frequency, and $f_o(v)$ the unperturbed velocity distribution.

In pursuing a solution of Eq.(1), we first note that a zero of the denominator, even though compensated by a zero of the numerator, corresponds to a maximum of $|N_{mt}|$ and therefore leads to slow convergence of the sum on m in (2). Since R/g may be in the range 100 to 1000, a maximum of the summand occurs at a large value of $m \simeq \pm(t\pi)R/g$. Values of m thousands of times larger than this value are required for good accuracy, and the situation gets worse as s and t increase. To avoid this problem, previous workers have restricted R/g to unrealistically small values. A standard technique to handle slowly-converging series is the Watson-Sommerfeld transformation, which works beautifully in this instance.[2] For the electromagnetic kernel (2) it yields $E_{st} = F_{st} + \Delta_s \delta_{st}$, where

$$F_{st} = -(g/R)(g/b)^2 (1+(-)^{s-t}) \sum_{i=1}^{\infty} \lambda_i \left((\lambda_i g/R)^2 + (s\pi)^2\right)^{-1} \times \left((\lambda_i g/R)^2 + (t\pi)^2\right)^{-1} \left(\cosh \pi\lambda_i + (-)^{s-1} \cosh(\pi - g/R)\lambda_i\right)/\sinh \pi\lambda_i \tag{7}$$

$$\Delta_s = (1+\delta_{so}) I_1(\Gamma_s b)/2\Gamma_s b I_o(\Gamma_s b) , \tag{8}$$

$$\lambda_i = R\left((j_{oi}/b)^2 - (\omega/c)^2\right)^{\frac{1}{2}} , \quad \Gamma_s = \left((s\pi)^2 - (\omega/c)^2\right)^{\frac{1}{2}}. \tag{9}$$

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In addition to eliminating the troublesome denominators, we have replaced Bessel functions by easily computed Bessel function zeros j_{oi}. The new sum F_{st} has cubic convergence, and may be computed accurately with 1000 terms, uniformly in the range of s and t required. A further bonus is that F_{st} vanishes relative to the diagonal matrix $\Delta_s\delta_{st}$ in the limit $b\to 0$. This is the diagonalization at small b/d mentioned above. Even at relatively large b/d, we may treat F_{st} as a perturbation, by the method of the sequel. A transformation of the series for S_{st} is not needed, since at most two terms of the series are important, as we shall see presently.

Elimination of the resonant mode and a perturbation method

Let us write (1) in the form

$$\overline{D}_s = Q_s \sum_{t=0}^{\infty} K_{st}\overline{D}_t + Q_s D_s^{(o)} , \qquad (10)$$

$$K_{st} = F_{st} + S_{st} , \qquad Q_s = R_s/(1 - R_s\Delta_s). \qquad (11)$$

Suppose that the frequency ω is near a cavity resonance in mode s = r. At small b, the resonance condition is $1 - R_r(\omega)\Delta_r(\omega) = 0$. Clearly, in developing a perturbation series we wish to avoid iteration of the resonance pole that occurs in Q_r. To that end we temporarily regard the resonant mode coefficient $\overline{D}_r$ as a fixed source term in Eq.(10), and solve the equations for the remaining $\overline{D}_s$, $s \neq r$. Those $\overline{D}_s$ are then linear functions of $\overline{D}_r$, which may be represented formally as

$$\overline{D}_s = \sum_{t\neq r} \{1 - QK\}^{-1}_{st} Q_t \left(K_{tr}\overline{D}_r + D_t^{(o)}\right), \ s \neq r, \qquad (12)$$

where $\{1 - QK\}$ is the matrix operator $(\delta_{st} - Q_s K_{st})$ restricted to the subspace of indices $s,t \neq r$. Now put s = r in Eq.(10), and substitute the result (12) for the non-resonant modes in the right-hand side of (10). Solving the resulting trivial equation for $\overline{D}_r$, we find

$$\overline{D}_r = \frac{R_r\left(D_r^{(o)} + \sum_{t,u\neq r} K_{rt}\{1-QK\}^{-1}_{tu}Q_u D_u^{(o)}\right)}{1 - R_r\left(\Delta_r + K_{rr} + \sum_{t,u\neq r} K_{rt}\{1-QK\}^{-1}_{tu}Q_u K_{ur}\right)} \qquad (13)$$

We are interested only in the denominator $D(\omega)$ of this expression. The growth time τ of an unstable perturbation is given by that zero of $D(\omega)$ in the upper-half ω-plane having the largest imaginary part: $1/\tau = (\mathrm{Im}\,\omega)_{max}$. Let G_t denote the solution of the resonance-free equations

$$\sum_{t\neq r} \{1-QK\}_{st} G_t = Q_s K_{sr} , \ s \neq r. \qquad (14)$$

Then

$$D = 1 - R_r\left(\Delta_r + K_{rr} + \sum_{t\neq r} K_{rt}G_t\right) , \qquad (15)$$

and a simple iterative solution of (14) yields the desired perturbation series

$$D = 1 - R_r\Big(\Delta_r + K_{rr} + \sum_{t\neq r} K_{rt}Q_t K_{tr} + \sum_{t,u\neq r} K_{rt}Q_t K_{tu}Q_u K_{ur} + \ldots..\Big). \qquad (16)$$

Of course, one is not restricted to a perturbative method; direct numerical solution of (14) is an efficient route to a precise solution, and is called for at large values of b/d. Direct solution of the original equation is not advisable, since its matrix is ill-conditioned in the region of interest and in fact is singular precisely at the zero of D that is sought.

Growth times of unstable perturbations

A closer look shows that the series (16) is still not ideal for computation of growth times, because the term S_{st} occurring in K_{st} has poles arising from the factor $1/(W_m^{-1}+\Theta_m)$ of (3). For instance, a square-step distribution $f_o(v)$ of full width Δv gives

$$\frac{1}{W_m^{-1}+\Theta_m} = \frac{\omega_p^2}{2\left((m\Delta v/2R)^2+\omega_p^2\Theta_m\right)^{\frac{1}{2}}}\left(\frac{1}{\omega-\omega_m^{(-)}} - \frac{1}{\omega-\omega_m^{(+)}}\right) \qquad (17)$$

$$\omega_m^{(\pm)} = m\Omega \pm \left((m\Delta v/2R)^2 + \omega_p^2\Theta_m\right)^{\frac{1}{2}} , \ \Omega = v_o/R \qquad (18)$$

A closely similar result is obtained for any acceptable distribution. For ω sufficiently close to one of these poles, the series (16) is clearly useless. The pole residues are extremely small, however, and it turns out that at the value of ω determining growth times one is far enough away from the poles so that the series is still useful, provided that the growth time is not too large. Nevertheless, better convergence and a more convenient formula is obtained by refining the method so as to avoid iteration of poles.

Numerical checks show that at most two terms in the sum (3) are at all important at any particular ω. That is true because the pole residues are small (10^{-10}/sec). Roughly speaking, ω is either close to one pole, in which case that pole gives nearly the entire value of the sum, or roughly equidistant from two poles, in which case those two dominate the sum. The minimum growth time occurs when parameters are such that $\mathrm{Re}\,\omega_r = \omega_m^{(-)}$ for some m, where ω_r is the cavity resonance zero defined by putting the self-interaction term equal to zero: $D(\omega_r) = 0$ for $S_{st} = 0$. In this situation, which may be achieved by adjusting the mean circulation frequency Ω, the pole at $\omega_m^{(-)}$ dominates the sum, and continues to dominate it for Ω in some neighborhood of the value for minimum growth time. In such a neighborhood we may write

$$S_{st} = N^{\dagger}_{sm}N_{mt}P_m(\omega-\omega_m^{(-)})^{-1}. \qquad (19)$$

Now S_{st} is a separable kernel (dyadic), which allows one to derive the following series in place of (16):

$$D = 1 - \sum_s Q_s\Big(V_s + \sum_{t\neq r} U_{st}Q_t V_t + \sum_{t,u\neq r} U_{st}Q_t U_{tu}Q_u V_u + \ldots\Big)N_{ms} , \qquad (20)$$

$$U_{st} = F_{st} - F_{sr}N_{mt}/N_{mr} , \qquad (21)$$

$$V_s = P_m N^{\dagger}_{sm}(\omega-\omega_m^{(-)})^{-1} + F_{sr}/N_{mr} . \qquad (22)$$

Of course, there is also a non-perturbative version of (20), analogous to (15). The pole now occurs only linearly, as a common factor of all terms in the bracket. We now assert that the expression (20) can be represented very precisely over the region of Ω of interest as

$$D(\omega) = D'_{e.m.}(\omega_r)(\omega-\omega_r) + \lambda(\omega-\omega_m^{(-)})^{-1} , \qquad (23)$$

where $D'_{e.m.}$ is the derivative of the purely electromagnetic part of D obtained by dropping the pole term (putting $P_m = 0$), and λ is a constant so small that two zeros of $D(\omega)$ are close to ω_r and $\omega_m^{(-)}$. The three constants $D'_{e.m.}$, ω_r, and λ are easily obtained numerically from (20), and the growth time τ is obtained from (23) by solving the quadratic equation $D(\omega) = 0$. In terms of $\mu^2 = \lambda/D'_{e.m.}$, which is almost real and positive, the answer is

$$\tau^{-1} = \mathrm{Im}\,\omega = \tfrac{1}{2}\left(\mathrm{Im}\,\omega_r + 2^{-\frac{1}{2}}\left((x^2+y^2)^{\frac{1}{2}} - x\right)^{\frac{1}{2}}\right) , \qquad (24)$$

$$x = \delta^2 - (\mathrm{Im}\,\omega_r)^2 - 4\mu^2, \quad y = 2\delta\,\mathrm{Im}\,\omega_r, \quad \delta = \mathrm{Re}\,\omega_r - \omega_m^{(-)} \qquad (25)$$

Now $\omega_m^{(-)}$, hence δ, is a linear function of the revolution frequency Ω, according to (18). The shortest growth time corresponds to $\delta = 0$, which is to say that the m-th harmonic of Ω is equal to the real part of the resonance frequency (plus a small shift given by the square root in (18)). Because of the sharpness of the resonance (high Q of a metal cavity), $\mathrm{Im}\,\omega$ has a sharp cusp at its maximum and decreases rapidly as $|\delta|$ increases. We illustrate for a square-step distribution and parameters of interest in ion beam fusion designs: 10 ampere beam of Hg^{++}; a=.03, b=.06, d=.3, g=.5, R=100, all in meters, and $\beta = v_o/c = .3$, $\Delta\beta/\beta = 3\times10^{-4}$, σ= cavity conductivity = $10^6 (m\Omega)^{-1}$. A rough evaluation of (20), subject to improvements now in progress, gave $\omega_r d/c = 2.412 - 9.916\times10^{-5} i$, $(\mu d/c)^2 = 9.64\times10^{-11}$. These numbers with (24) give the results in the table, where $\Delta\Omega$ represents the deviation of Ω from the value giving the shortest growth time, and τ is in milliseconds.

$(\Delta\Omega/\Omega)\times10^5$	0	2	4	6	8	10	12
τ(ms)	1.05	1.29	2.01	3.22	4.93	7.13	9.84

As $\Delta\Omega$ is increased beyond the values in the table, τ eventually begins to decrease and reaches its minimum value again as the pole at $\omega_{m-1}^{(-)}$ comes to the position $\mathrm{Re}\,\omega_r$. This occurs approximately at $\Delta\Omega/\Omega = 1/m = 1/2680 = 37\times10^{-5}$; the growth time is periodic in $\Delta\Omega/\Omega$ with period 1/m. In a region midway between two minima of τ, the formula (24) is not accurate, since two poles, at $\omega_{m-1}^{(-)}$ and $\omega_m^{(-)}$, contribute. We have not yet computed τ in the midregion, but it appears that the growth time is greater than 10 times minimum in perhaps 30% of Ω-space, and greater than 2 times minimum for about 80% of Ω-space.

There is also a zero associated with $\omega_m^{(+)}$, but it is in the lower half-plane, because the sign of the pole residue is opposite to that of $\omega_m^{(-)}$. An anomalous situation arises if parameters are adjusted so that $\omega_{m-1}^{(+)} = \omega_m^{(-)}$; this may be done with reasonable parameters. Since the pole residues vary slowly with m, and (+) and (-) residues have opposite signs, there is a near cancellation of one pole by the other in this case. Then the growth time is unusually long, even if $\omega_m^{(-)} = \mathrm{Re}\,\omega_r$. Perhaps this curious effect deserves further thought.

References

1. E. Keil and B. Zotter, Particle Accelerators 3, 11-20, (1972).

2. A Watson-Sommerfeld transformation was applied in the present problem, but in a different way, by M. Month and R. Peierls, Nuclear Instruments and Methods 137, 299-318, (1976). Month and Peierls applied the transformation to the solution of the equations, a priori an unknown function. They therefore had to use unproved analyticity properties of the function transformed. In our case, the function is known, and the transformation is rigorously justified.

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