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ON THE COUPLING IMPEDANCE OF A RESONANT CAVITY

Toshio SUZUKI

National Laboratory for High Energy Physics

Summary

Longitudinal coupling impedance of a resonant cavity is studied by the normal mode analysis using the Coulomb gauge. The vector potential giving a solenoidal electric field leads to a resonant form described by a parallel LCR circuit. The coupling impedance at resonance is equal to the shunt impedance of the cavity. The scalar potential giving an irrotational electric field results in a non-resonant and purely imaginary coupling impedance. Since this latter impedance plays no role in power consumption, it has no relation to the shunt impedance of a cavity. Explicit expressions of the coupling impedance are given for a pill-box cavity considered by Keil et al.[1] It is shown that their result on higher-order mode loss is derived from the real part of the coupling impedance considered here.

§1. Introduction

It seems to be known for some time that the coupling impedance of a resonant cavity which appears in the theory of collective beam instabilities is equivalent to the shunt impedance of a cavity. In a classical paper by Neil and Sessler[2], the coupling impedance is put equal to the shunt impedance without any remark. Later, Schnell[3] proved the equivalence of the coupling impedance and the shunt impedance of a cavity from the consideration on the energy dissipation in a cavity.

Though Schnell's derivation is simple and intuitive, two important points are missing which will be clarified in this paper. First of all, the resonant behavior of the coupling impedance, which is often explained by resorting to a parallel LCR circuit, is not clear. Secondly, since his derivation is based on energy dissipation, the out-of-phase or purely imaginary coupling impedance cannot be dealt with. Indeed, it is shown in this paper that the irrotational part of the electric field gives rise to a purely imaginary coupling impedance.

In order to make the situation transparent, we deal with the electromagnetic field directly by normal mode analysis using the Coulomb gauge.[4,5] As an application of the present approach, we give explicit expressions for the coupling impedance of a pill-box cavity considered by Keil et al.[1] We will also discuss their result on higher order mode loss in terms of the coupling impedance derived here.

§2. Normal Mode Analysis

In this section, we review the normal mode analysis for the electromagnetic fields in a resonant cavity.[4,5,6] We use the Coulomb gauge. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are derived from the vector potential $\mathbf{A}$ and the scalar potential ϕ

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \qquad \mathbf{B} = \nabla\times\mathbf{A}, \tag{1}$$

where c is the velocity of light. The potentials satisfy the equations

$$\nabla^2\phi = -4\pi\rho, \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}\nabla\phi, \tag{2}$$

where ρ is the charge density and $\mathbf{j}$ is the current density. The boundary conditions are, if we assume a perfectly contucting surface,

$$\mathbf{A}\times\mathbf{n} = 0, \quad \phi = 0, \quad \text{(on the surface)} \tag{3}$$

where $\mathbf{n}$ is the normal extending inside from the surface of the cavity.

We introduce the normal mode fields $\mathbf{A}_s$ and ϕ_s, which satisfy

$$\nabla^2\mathbf{A}_s + \frac{\omega_s^2}{c^2}\mathbf{A}_s = 0, \qquad \nabla^2\phi_s + \frac{\omega_s^2}{c^2}\phi_s = 0, \tag{4}$$

where ω_s denotes the eigenfrequency of the modes. The fields $\mathbf{A}_s$ and ϕ_s satisfy the boundary conditions (3) and are normalized such that

$$\int_V \mathbf{A}_s \mathbf{A}_{s'}\, dV = 4\pi c^2 \delta_{ss'}, \qquad \int_V \phi_s \phi_{s'}\, dV = 4\pi c^2 \delta_{ss'}, \tag{5}$$

where the integral is done over the inside volume V of the cavity. We expand the potentials in terms of the normal modes

$$\mathbf{A}(\mathbf{r},t) = \sum_s q_s(t)\,\mathbf{A}_s(\mathbf{r}), \qquad \phi(\mathbf{r},t) = \sum_s r_s(t)\,\phi_s(\mathbf{r}). \tag{6}$$

Then, we obtain the following equations for the coefficients $q_s(t)$ and $r_s(t)$,

$$\ddot{q}_s(t) + \omega_s^2 q_s(t) = \frac{1}{c}\int_V \mathbf{j}(\mathbf{r},t)\,\mathbf{A}_s(\mathbf{r})\,dV, \tag{7}$$

$$r_s(t) = \frac{1}{\omega_s^2}\int \rho(\mathbf{r},t)\,\phi_s(\mathbf{r})\,dV. \tag{8}$$

We note that the equations for the vector and scalar potentials are decoupled. We also note that the vector potential gives a solenoidal part of the electric field and the scalar potential gives an irrotational part of the electric field in the nomenclature of Slater.[6]

Now, we consider the effect of a finite conductivity of the cavity wall. Since the vector potential gives the solenoidal part of the electric field, the method of Slater[6] gives instead of (7) the following equation for $q_s(t)$ (we assume the time dependence $e^{-i\omega t}$ instead of $e^{j\omega t}$ in this paper)

$$\ddot{q}_s(t) + (1-i)\frac{\omega_s}{Q_s}\dot{q}_s(t) + \omega_s^2 q_s(t) = \frac{1}{c}\int_V \mathbf{j}(\mathbf{r},t)\,\mathbf{A}_s(\mathbf{r})\,dV \tag{9}$$

Here, the quality factor Q_s is given as[6]

$$\frac{1}{Q_s} = \frac{\delta}{2}\int_S H_s^2\, dS \Big/ \int_V H_s^2\, dV \tag{10}$$

where δ is the skin depth of the wall, H_s denotes the magnetic field intensity and the integral is over the surface S or over the volume V. Since the scalar potential does not give a magnetic field, no change will occur in the equation (8) for $r_s(t)$.

§3. Longitudinal Coupling Impedance of a Cavity

According to the definition of the coupling impedance, we study the electromagnetic field excited in a cavity by harmonic current and charge densities

$$j_z(\mathbf{r},t) = \frac{I}{2\pi r}\delta(r)\, e^{i(kz-\omega t)}, \qquad \rho(\mathbf{r},t) = \frac{\lambda}{2\pi r}\delta(r)\, e^{i(kz-\omega t)}, \tag{11}$$

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$$I=\lambda v,$$
$$\omega=kv.$$

Then, from eq. (9), we get

$$q_s(t)=\frac{M_s I/c}{\omega_s^2-\omega^2-(1+i)\frac{\omega\omega_s}{Q_s}}e^{-i\omega t}, \qquad (12)$$

$$M_s=\int A_{zs}(0,0,z)e^{ikz}dz, \qquad (13)$$

and, from eq. (8), we get

$$r_s(t)=\frac{\lambda\alpha_s}{\omega_s^2}e^{-i\omega t}, \qquad (14)$$

$$\alpha_s=\int \varphi_s(0,0,z)e^{ikz}dz \qquad (15)$$

The induced longitudinal electric field is respectively given by

$$E_z=-\frac{\dot{q}_s}{c}A_{zs}, \qquad (16)$$

and

$$E_z=-r_s\frac{\partial\varphi_s}{\partial z}. \qquad (17)$$

We resolve the standing electric field wave into travelling waves and keep only that part which travels in phase with the rf current and charge waves. Then, the voltage per turn V due to these electric fields is given by

$$V=\int E_z e^{-ikz}dz \qquad (18)$$

Using the definition of the coupling impedance

$$V=-ZI \qquad (19)$$

we obtain

$$Z(\omega)=\sum_s\frac{-i\omega|M_s|^2/c^2}{\omega_s^2-\omega^2-(1+i)\frac{\omega\omega_s}{Q_s}}, \qquad (20)$$

for the solenoidal part and

$$Z(\omega)=\sum_s\frac{i\omega|\alpha_s|^2}{v^2\omega_s^2}, \qquad (21)$$

for the irrotational part. Eq. (20) is of a resonant type and described by a parallel LCR resonant circuit. Eq. (21) is purely imaginary and non-resonant.

Referring to eq. (20), which is the coupling impedance due to the solenoidal electric field, the resonance frequency ω_r is given by

$$\omega_s^2-\omega_r^2-\frac{\omega_r\omega_s}{Q_s}=0. \qquad (22)$$

At resonance, the coupling impedance is given by

$$Z(\omega_r)=\frac{|M_s|^2Q_s}{\omega_s c^2}. \qquad (23)$$

Now,

$$M_s=\int A_{zs}(0,0,z)e^{ikz}dz$$
$$=-\frac{c}{\dot{q}_s}\int E_z e^{ikz}dz. \qquad (24)$$

$\left|\int E_z e^{ikz}dz\right|=V$ is the accelerating voltage including the transit time factor and the stored energy W in the cavity is given by[5]

$$W=\frac{1}{2}[\dot{q}_s^2+\omega_s^2q_s^2]=\frac{1}{2}|\dot{q}_s|^2. \qquad (25)$$

Using the relation $Q_s=\frac{\omega_s W}{P}$, we finally obtain

$$Z(\omega_r)=\frac{V^2}{2P}, \qquad (26)$$

which is equal to the shunt impedance of the cavity. Thus, the equivalence of the coupling impedance and the shunt impedance is proved. The purely imaginary coupling impedance due to the irrotational electric field, however, is not related to the shunt impedance.

§4. Coupling Impedance of a Pill-Box Cavity and High Order Mode Loss

Now, we consider a pill-box cavity considered by Keil et al.[1] It is shown in Fig. 1. The side wall is assumed to have a finite conductivity and the walls perpendicular to the beam are assumed to be infinitely conducting. The normal mode fields for the TM-mode vector potential and the scalar potential are

$$A_r=\frac{p\pi}{g}\sqrt{\frac{8c^4}{b^2gJ_1^2(\nu_s)\omega_s^2(1+\delta_{p0})}}J_1\left(\nu_s\frac{r}{b}\right)\sin\frac{p\pi}{g}z,$$

$$A_\theta=0, \qquad (27)$$

$$A_z=\sqrt{\frac{8c^4\nu_s^2}{b^4gJ_1^2(\nu_s)\omega_s^2(1+\delta_{p0})}}J_0\left(\nu_s\frac{r}{b}\right)\cos\frac{p\pi}{g}z,$$

$$\phi=\sqrt{\frac{8c^2}{gb^2J_1^2(\nu_s)}}J_0\left(\nu_s\frac{r}{b}\right)\sin\frac{p\pi}{g}z,$$

where

$$\left(\frac{\omega_s}{c}\right)^2=\left(\frac{\nu_s}{b}\right)^2+\left(\frac{p\pi}{g}\right)^2 \qquad (28)$$

p is an integer and ν_s is the s-th zero of the Bessel function $J_0(x)$. Then,

$$|M_s|^2=\frac{2\left(\frac{\omega}{v}\right)^2}{\left[\left(\frac{\omega}{v}\right)^2-\left(\frac{p\pi}{g}\right)^2\right]^2}\frac{8c^4\nu_s^2}{b^4gJ_1^2(\nu_s)\omega_s^2(1+\delta_{p0})}\times$$
$$\times\left[1-(-1)^p\cos\frac{\omega}{v}g\right], \qquad (29)$$

$$\frac{1}{Q_s}=\frac{\delta}{b}, \qquad (30)$$

and

$$|\alpha_s|^2=\frac{\left(\frac{p\pi}{g}\right)^2}{2\left[\left(\frac{\omega}{v}\right)^2-\left(\frac{p\pi}{g}\right)^2\right]^2}\cdot\frac{8c^2}{gb^2J_1^2(\nu_s)}\left[1-(-1)^p\cos\frac{\omega}{v}g\right]. \qquad (31)$$

Eqs (29), (30), (31) and the expressions (20), (21) give the coupling impedance of the pill-box cavity.

The higher order mode loss per turn ΔU in a stationary multi-turn case is given by

$$\Delta U=2I_{av}^2\sum_{n=1}^{\infty}\mathrm{Re}\,Z(n\omega_0)e^{-n^2\omega_0^2\sigma^2}, \qquad (32)$$

where I_{av} is the average currnet per bunch, ω_0 is the revolution frequency, σ is the bunch length and Re denotes the real part. It can be shown by some algebra that the result of Keil et al[1] for the higher order mode loss is equal to eq. (32) if we take the coupling impedance $Z(n\omega_0)$ derived here. The coupling impedance due to the irrotational electric field is purely imaginary and does not contribute to energy loss.

Reference

1) E. Keil, C. Pellegrini, A. Turrin and A.M. Sessler: Nucl. Instrum, Meth. 127 (1975) 475.
2) V.K. Neil and A.M. Sessler: Rev. Sci. Instrum. 32 (1961) 256.
3) W. Schnell: CERN/ISR-RF/70-7 (1970).
4) E.V. Condon: J. Appl. Phys. 12 (1941) 129.
5) P.L. Morton and V.K. Neil: Symp. Electron Ring Accelerators, UCRL 18103 (1968) p.365.
6) J.C. Slater: "Microwave Electronics", Chap. 4 (D. Van Nostrand, 1950).

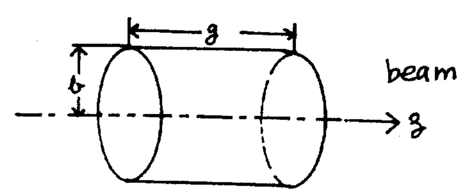

Fig. 1