IEEE Transactions on Nuclear Science, Vol. NS-28, No. 3, June 1981

EDGE FOCUSING*

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Abstract

Beam transport matrix elements describing the linearly falling fringe field of a combined function bending magnet are expanded in powers of the fringe field length by iteratively solving the integral form of Hill's equation. The method is applicable to any linear optical element with variable focusing strength along the reference orbit. Results for the vertical and horizontal focal lengths agree with previous calculations[1] for a zero gradient magnet and an added correction to the dispersion is found for this case. A correction to the fringe field gradient peculiar to a combined-function magnet with strong edge focusing is also found. The influence of edge focusing on the tunes and chromaticities of the NSLS rings is described. The improved chromaticity calculation for the booster was of particular interest since this ring has bending magnets with poletips shaped to achieve small positive chromaticities.

Transfer Matrix for Arbitrary Linear Focusing Element

A systematic approximation to the transport matrix of an arbitrary linear element may be obtained by writing an integral equation equivalent to Hill's equation. We start with the tautology

$$x(s) = x_o + x'_o \cdot (s-s_o) + \int_{s_o}^{s} \int_{s_o}^{s_1} x''(s_2)ds_2 ds_1.$$

where x_o, x'_o are constants.

We require $x''(s) + K(s) \cdot x(s) = 0$ so

$$x(s) = x_o + x'_o \cdot (s-s_o) - \int_{s_o}^{s} \int_{s_o}^{s_1} K(s_2)x(s_2)ds_2 ds_1.$$

We do the substitution iteratively

$$x(s) = x_o + x'_o \quad (s-s_o) - \int_{s_o}^{s} ds_1 \int_{s_o}^{s} ds_2 K(s_2) \cdot$$

$$\left[x_o + x'_o \cdot (s_2-s_o) - \int_{s_o}^{s_2} ds_3 \int_{s_o}^{s_3} ds_4 K(s_4)\left[x_o + x'_o \cdot (s_4-s_o) - \ldots\right]\right]$$

We group separately the coefficients of x_o and x'_o to find the elements A, B in the top row of the 2x2 transport matrix

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} A(s,s_o) & B(s,s_o) \\ C(s,s_o) & D(s,s_o) \end{bmatrix} \cdot \begin{bmatrix} x_o \\ x'_o \end{bmatrix}$$

$$A = 1 - \int_{s_o}^{s} ds_1 \int_{s_o}^{s_1} ds_2 K(s_2) + \int_{s_o}^{s} ds_1 \int_{s_o}^{s_1} ds_2 K(s_2) \int_{s_o}^{s_2} ds_3 \int_{s_o}^{s_3} ds_4 K(s_4) - \ldots \tag{1}$$

$$B = (s-s_o) - \int_{s_o}^{s} ds_1 \int_{s_o}^{s_1} ds_2 K(s_2) \cdot (s_2-s_o) + \int_{s_o}^{s} ds_1 \int_{s_o}^{s_1} ds_2 K(s_2) \cdot \int_{s_o}^{s_2} ds_3 \int_{s_o}^{s_3} ds_4 K(s_4) \cdot (s_4-s_o) - \ldots \tag{2}$$

$$C(s,s_o) = \frac{d}{ds} A(s,s_o)$$
$$D(s,s_o) = \frac{d}{ds} B(s,s_o) \tag{3}$$

It is easy to verify that the above expressions are correct for the case K(s) = constant. The technique described above is commonly generalized to approximate the effect of nonlinear focusing terms, but is it a useful way to systematically approximate linear focusing elements in which K(s) varies rapidly with s, such as edge focusing or wiggler magnets.

Here the technique will be applied to a fringe field derived from the magnetic scalar potential.

$$\phi(a,y,z) = \frac{-B_o yz}{\Delta} \left(1 + \frac{G}{B_o} a\right)$$

The $\{a,y,z\}$ coordinate system, illustrated in Fig. 1, has axes parallel to the symmetry planes of the magnet. From the potential,

$$B_y = \frac{B_o z}{\Delta} \left(1 + \frac{G}{B_o} a\right) \quad \text{for } 0<z<\Delta, \tag{4}$$

$B_y = B_o + G \cdot a$ for $z>\Delta$. and

$B_y = 0$ for $z<0$.

Consider a particle approaching the magnet end along the reference orbit shown in Fig. 1. Before entering the magnet, the particle trajectory makes an angle θ_o with the z axis. We may express the angle between the reference orbit and the z axis in terms of the pathlength s along the reference orbit:

$$\theta(s)=\theta_o-\phi(s);\quad \phi(s)=\int \frac{B_y(s)ds}{B\rho} = \int \frac{B_y(a,y,z)\ dz}{B\rho \cos(\theta(s(z)))} \tag{5}$$

We substitute (4) into (5) above and approximate $G \cdot a$ by $G \cdot z \tan\theta_o$. We expand $\frac{1}{\cos(\theta(s))}$ around $\theta(s) = \theta_o$, to order ϕ^2. By iterative substitution we can

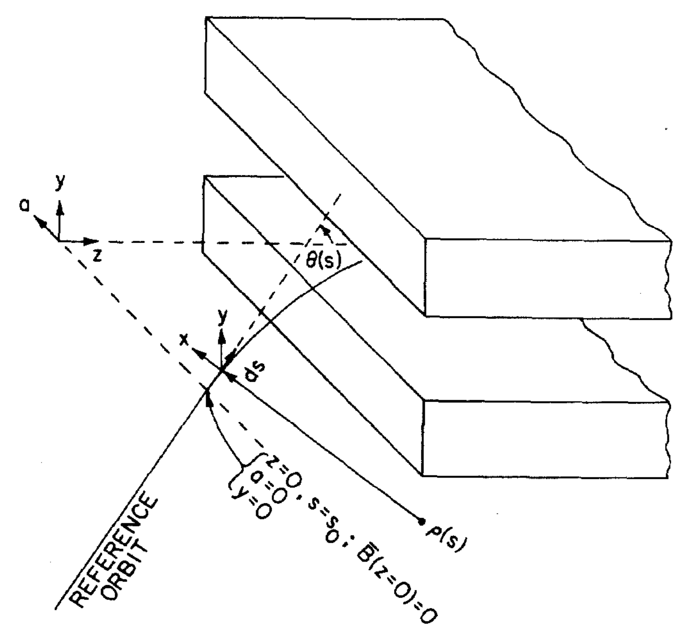

Figure 1. Coordinate Systems

*Research supported by the U.S. Department of Energy.

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approximate to order $\left(\frac{B_o}{B\rho}\right)^2$ and $\left(\frac{G}{B\rho}\right)$:

$$\phi = \frac{G}{B\rho\Delta}\,\frac{\sin\theta_o}{\cos^2\theta_o}\,\frac{2z^3}{3!} + \frac{B_o}{B\rho\Delta}\,\frac{1}{\cos\theta_o}\,\frac{z^2}{2!} - \left(\frac{B_o}{B\rho\Delta}\right)^2 \frac{\sin\theta_o}{\cos^3\theta_o}\,\frac{3z^4}{4!} \tag{6}$$

We can now calculate the path length of the reference orbit in the fringe field

$$\delta = \int_o^\Delta \frac{dz}{\cos(\theta_o - \phi(z))}$$

or again expanding $\frac{1}{\cos\theta}$ around θ_o and using (6):

$$\delta = \Delta\left[\frac{1}{\cos\theta_o} - \frac{B_o\Delta}{B\rho}\,\frac{\sin\theta_o}{\cos^3\theta_o}\,\frac{1}{3!} + \left(\frac{B_o\Delta}{B\rho}\right)^2 \frac{1+2\sin^2\theta_o}{\cos^5\theta_o}\,\frac{3}{5!} - \frac{G\Delta^2}{B\rho}\,\frac{\sin^2\theta_o}{\cos^4\theta_o}\,\frac{2}{4!}\right] \tag{7}$$

We must write the derivatives of B_y in a curvilinear coordinate system $\{x,y,s\}$ associated with the reference orbit (Figure 1)

$$K_y(s) = \frac{+1}{B\rho}\frac{\partial B_x}{\partial y} = \frac{-1}{B\rho}\frac{\partial B_y(s)}{\partial x} = \frac{-1}{B\rho}\left(\frac{\partial B_y}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial B_y}{\partial z}\frac{\partial z}{\partial x}\right) \tag{8}$$

$$\frac{\partial a}{\partial x} = \cos\theta(s), \quad \frac{\partial z}{\partial x} = -\sin\theta(s) \tag{9}$$

We substitute (9) into (8), expand the sines and cosines to order ϕ and substitute

$$\phi = \frac{B_o}{B\rho\Delta}\,\frac{1}{\cos\theta_o}\,\frac{z^2}{2}, \quad z^2 = s^2\cos^2\theta_o$$

to get $K_y(s)$ to order $\left(\frac{B_o}{B\rho}\right)^2$ and $\frac{G}{B\rho}$:

$$K_y(s) = -\frac{Gs}{B\rho\Delta}\cos 2\theta_o + \frac{B_o}{B\rho\Delta}\sin\theta_o - \left(\frac{B_o}{B\rho\Delta}\right)^2 \frac{s^2}{2!}\cos^2\theta_o \tag{10}$$

We may now substitute (10) into the formulas for the transport matrix elements (2), (3), and make all integrals dimensionless[3] by replacing

$$\int_{s_o}^{s} ds_1 \text{ by } \delta\int_o^1 du_1, \quad s_1 = \delta u_1 + s_o.$$

The result is

$$A = 1 \pm \frac{\delta}{\Delta}\left[(g)\frac{\cos 2\theta_o}{3!} - (b)\frac{\sin\theta_o}{2!} + \frac{\delta}{\Delta}(b^2)\frac{\cos^2\theta_o}{4!}\right] \tag{11}$$

$$+ \left(\frac{\delta}{\Delta}\right)^2 (b^2)\frac{\sin^2\theta_o}{4!} \begin{cases} +0 \\ -\left(\frac{\delta}{\Delta}\right)^2 (b^2)\dfrac{2\cos^2\theta_o}{4!} \end{cases}$$

$$B = \delta \pm \frac{\delta^2}{\Delta}\left[(g)\frac{2\cos 2\theta_o}{4!} - (b)\frac{\sin\theta_o}{3!} + \frac{\delta}{\Delta}(b^2)\frac{3\cos^2\theta_o}{5!}\right] \tag{12}$$

$$+ \frac{\delta^3}{\Delta^2}(b^2)\frac{\sin^2\theta_o}{5!} \begin{cases} +0 \\ -\dfrac{\delta^3}{\Delta^2}(b^2)\dfrac{6\cos^2\theta_o}{5!} \end{cases}$$

$$C = \pm\frac{1}{\Delta}\left[(g)\frac{\cos 2\theta_o}{2!} - (b)\sin\theta_o + \frac{\delta}{\Delta}(b^2)\frac{\cos^2\theta_o}{3!}\right] \tag{13}$$

$$+ \frac{\delta}{\Delta^2}(b^2)\frac{\sin^2\theta_o}{3!} \begin{cases} +0 \\ -\dfrac{\delta}{\Delta^2}(b^2)\dfrac{2\cos^2\theta_o}{3!} \end{cases}$$

$$D = 1 \pm \frac{\delta}{\Delta}\left[(g)\frac{2\cos 2\theta_o}{3!} - (b)\frac{\sin\theta_o}{2!} + \frac{\delta}{\Delta}(b^2)\frac{3\cos^2\theta_o}{4!}\right] \tag{14}$$

$$+ \left(\frac{\delta}{\Delta}\right)^2 (b^2)\frac{\sin^2\theta_o}{4!} \begin{cases} +0 \\ -\left(\dfrac{\delta}{\Delta}\right)^2 (b^2)\dfrac{6\cos^2\theta_o}{4!} \end{cases}$$

where $g = \frac{G\delta^2}{B\rho}$, $b = \frac{B_o\delta}{B\rho}$

Here the upper sign is for the vertical motion and the lower is for the horizontal motion. The last term in each expression is the contribution to horizontal focusing from the orbit curvature:

$$K_x = -K_y + \frac{1}{\rho^2(s)}.$$

One may verify that the determinants for the 2x2 submatrices equal $1+O(b^3)$. Exact unimodularity should be enforced artificially be defining

$$A = \frac{1+BC}{D}$$

which adds corrections of order b^3, $b\,g$, g^2 and higher to A.

The dispersion $\eta(s)$ may be calculated by the method used for the betatron transfer matrix, since

$$\eta''(s) + K_x\,\eta(s) = \frac{1}{\rho}$$

$$\eta(s_o)=0,\ \eta'(s_o)=0;\ \eta(s) = \int_{s_o}^{s}\int_{s_o}^{s_1}\left[\frac{1}{\rho(s_2)} - \eta(s_2)K_x(s_2)\right]ds_2\,ds_1 \tag{15}$$

For the fringe field assumed earlier,

$$\eta = \delta\left[\frac{G\delta^3}{B\rho\Delta}\sin\theta_o\cos\theta_o\,\frac{2}{4!} + \frac{B_o\delta^2}{B\rho\Delta}\cos\theta_o\,\frac{1}{3!} + \left(\frac{B_o\delta^2}{B\rho\Delta}\right)^2 \frac{2\sin\theta_o\cos\theta_o}{5!}\right] \tag{16}$$

$$\eta' = \frac{G\delta^3}{B\rho\Delta}\,\frac{2\sin\theta_o\cos\theta_o}{3!} + \frac{B\delta^2}{B\rho\Delta}\,\frac{\cos\theta_o}{2!} + \left(\frac{B_o\delta^2}{B\rho\Delta}\right)^2 \frac{2\sin\theta_o\cos\theta_o}{4!} \tag{17}$$

Finally, the change in path length is

$$\frac{ds}{d\left(\frac{\Delta p}{p}\right)} = \left[\frac{B_o\Delta}{B\rho}\,\frac{\sin\theta_o}{\cos^3\theta_o}\,\frac{1}{3!} - \left(\frac{B_o\Delta}{B\rho}\right)^2 \frac{1+2\sin^2\theta_o}{\cos^5\theta_o}\,\frac{6}{5!} + \frac{G\Delta^2}{B\rho}\,\frac{\sin^2\theta_o}{\cos^4\theta_o}\,\frac{2}{4!}\right]\cdot\Delta \tag{18}$$

Figure 1 and Equation (10) show that we have placed the origin of the (a,y,z) coordinate system at the point where the reference orbit leaves the field-free region. The reference orbit enters the body of the magnet ($z \geq \Delta$) at the point where the field is

$$B_c = B_o + G \int_o^\Delta \tan\theta(z)\, dz$$

and the gradient is $G_c = \frac{\partial B_y}{\partial x} = G\cos\theta(z=\Delta)$. Since it is customary to express transport matrix elements in terms of B_c, G_c, we may substitute $B_o \simeq B_c - G\,\Delta\tan\theta_o$, $G \simeq \frac{G_c}{\cos\theta_o}$ in Equations (11) - (18). We note that $\frac{\partial B_y}{\partial x}$

Changes abruptly at $z = \Delta$ because $\frac{\partial B_y}{\partial z}$ changes to zero inside the magnet. In terms of B_c and G_c, (13) becomes

$$C_x = -\frac{1}{f_x} \simeq \frac{B_c}{B\rho}\tan\theta_o - \left(\frac{B_c}{B\rho}\right)^2 \frac{\Delta}{2\cos\theta_o} - \frac{G_c}{\cos^3\theta_o}\,\frac{\Delta}{2} \tag{19}$$

$$C_y = -\frac{1}{f_y} \simeq \frac{B_c}{B\rho}\tan\theta_o - \left(\frac{B_c\Delta}{B\rho}\right)^2 \frac{1+\sin^2\theta_o}{6\Delta\cos^3\theta_o} + \frac{G_c}{\cos^3\theta_o}\,\frac{\Delta}{2B\rho} \tag{20}$$

for $G_c = 0$, we find (20) agrees with previous calculations of C_y.[1 2] We also find that C_x agrees with Enge's result: The horizontal focal length of the extended fringe field is, to order $\left(\frac{B_c}{B\rho}\right)^2$, equal to that of a "hard edge" (Δ=0 edge effect) magnet of constant field B_c, which bends the reference orbit through the same angle as the extended fringe field. From (6) the "hard edge" magnet should have length $\Delta/(2\cos\theta_o)$ measured along the reference orbit.

Equations (16), (17), however, contradict Enge's assertion that the dispersion caused by an extended fringe field equals that of the "hard edge" magnet to order $\left(\frac{B}{B\rho}\right)^2$. We recover Enge's result only if we neglect the edge-focusing contribution to $K_x(s)$ in solving (15). Since $K_x = \frac{B_o}{\Delta B\rho}\sin\theta_o$, the case $\theta_o = 0$ agrees with Enge.

One would normally replace the extended fringe field of a combined-function magnet by the hard edge magnet with length $\Delta/2\cos\theta_o$ and gradient G_c. The lowest order contribution to the focal length would be given by

$$C^y_x \rightarrow C^y_x \pm \frac{G_c}{B\rho}\,\frac{\Delta}{2\cos\theta_o}$$

Comparing with (19), (20), we see that this underestimates the fringe field gradient by a factor $\sec^2\theta_o$.

The importance of these corrections to edge focusing depends on the optical system to which they are added. To assess the effect on the NSLS rings, the program SYNCH[4] was used to compute tunes and chromaticities of the 700 MeV storage ring and booster. Table 1 shows the results for the 700 MeV storage ring. The leftmost column lists the fringe field length inmeters. Under the heading "THICK" appear the tunes and chromaticities (labeled ν and ξ respectively) calculated using the edge focusing matrices described here. Under the heading "THIN" are the same quantities, computed using Enge's edge focusing matrices. Both cases show a very large vertical tune shift as Δ increases from 0 to 0.3 meters. The disagreement between the two edge focusing approximations is no greater than one would expect for an order $\left(\frac{B}{B\rho}\right)^2$ calculation. Table II shows the same information for the booster synchrotron. The downward shift of horizontal tune in the thick lens case is caused by the $\sec^2\theta_o$ correction to the gradient term in the fringe field. The difference in horizontal chromaticity is numerically significant and important because the main quad and sextupole circuits do not allow independent control of ν_x, ν_y, ξ_x and ξ_y. The design chromaticity of the booster is $\xi_{x,y} = +0.5$ so $\Delta\xi_x = -0.4$ caused by thick lens effects cannot be neglected.

The author gratefully thanks E. Courant and E. Bozoki for their help in modifying SYNCH to include extra edge focusing matrices, and Sam Krinsky for many useful discussions.

References

1. Harald A. Enge, in Focusing of Charged Particles, A. Septier, Editor, Vol. 2, pp. 239-248, Academic Press, New York and London, 1967.
2. K. Brown, "A First- and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers," SLAC-75 (1967), pp. 71-76.
3. S. Krinsky, NSLS Technical Note, in preparation.
4. A. A. Garren and A. S. Kenney, LBL Internal Reports 11/68 and 2/74.

Table 1. NSLS 700 MeV Storage Ring.

		THIN		THICK	
Δ		x	y	x	y
0.0m	ν	3.32	1.32	3.32	1.32
	ξ	-4.1	-5.4	-4.1	-5.4
0.1m	ν	3.32	1.2231	3.3194	1.2236
	ξ	-4.1	-5.5	-4.1	-5.4
0.2m	ν	3.32	1.1207	3.3177	1.1227
	ξ	-4.1	-5.6	-4.1	-5.4
0.3m	ν	3.32	1.0111	3.3149	1.0159
	ξ	-4.1	-5.8	-4.1	-5.6

Table 2. NSLS Booster

		THIN		THICK	
Δ		x	y	x	y
0.0m	ν	2.4295	2.3903	2.4295	2.3903
	ξ	-3.0	-4.3	-3.0	-4.3
0.1m	ν	2.4295	2.3630	2.4273	2.3617
	ξ	-3.0	-4.2	-2.9	-4.3
0.2m	ν	2.4295	2.3358	2.4204	2.3311
	ξ	-3.0	-4.1	-2.8	-4.3
0.3m	ν	2.4295	2.3088	2.4082	2.2996
	ξ	-3.0	-4.0	-2.6	-4.2