CHARGED PARTICLE BEAM TRANSPORT USING LIE ALBEGRAIC METHODS*

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Abstract

Methods have recently been developed enabling one to describe charged particle beam transport in terms of a Lie algebraic representation. We show how this formulation may be applied to the problem of computing transfer maps (including aberrations) for the beam-line elements common in accelerator physics. In addition, explicit calculations are carried out for a variety of common elements including drift sections, static magnetic dipole, quadrupole, and sextupole lengths, and dynamic sectons such as bunchers. The resultant mappings are displayed, and provide a succinct basis for efficient numerical computation of charged particle beam behavior during transport.

Introduction

Since their introduction over twenty years ago[1], matrix methods have proven extremely useful in the description of charged particle beam behavior. In principle, matrix methods can be used to treat aberrations of arbitrarily large order. However, their application to aberrations beyond the lowest order requires the storage and manipulation of quite large matrices. For example, if current matrix methods were extended to work through the general third order at fixed momentum, the use of 34x34 matrices would be required. If chomatic effects were included as well, 55x55 matrices would be required.

In working with expansions beyond second order, it appears to be useful to take into account the Hamiltonian character of the equations of motion. Because the equations of motion are Hamiltonian in form the relation between incoming and outgoing beam coordinates must be a canonical transformation, and consequently many of the matrix elements required for a matrix method are in fact related by a complicated set of linear and nonlinear identities. This redundancy can be conveniently avoided by the use of Lie algebraic methods.[2] For example, again working through third order, the use of Lie algebraic methods would require the storage of 65 parameters in the fixed momentum case, and 120 parameters if all chromatic effects are included. Storage requirements over matrix methods are thus reduced by factors of approximately 18 and 25 respectively.

The purpose of this note is to outline the use of Lie algebraic methods. The tools necessary for re-presenting canonical transformations in terms of a Poisson bracket Lie algebra will be described, and illustrative calculations for quadrupole and sextupole elements presented. Results for some other common beam line elements will also be given, and a comparison to matrix methods made. Finally we indicate how such methods may be extended to arbitrarily high orders.

Lie Algebraic Tools

Consider a particle of rest mass m_0 and charge q moving through a beam line element. Let (x,y) denote its transverse displacement from the design orbit and

parameterize the motion by the particle's displacement z along the design orbit. The trajectory equations with z as an independent variable are then canonical in form, and can be obtained from the Hamiltonian[3]:

$$K(x\ y\ t; p_x\ p_y\ p_t) \equiv -c^{-1}\{[p_t + q\phi(x,y,t)]^2 - m_0^2 c^4$$
$$-c^2[p_x - qA_x(x,y,t)]^2$$
$$-c^2[p_y - qA_y(x,y,t)]^2\}^{1/2} - qA_z(x,y,t)$$

Here p_t is a momentum conjugate to the time, and $\vec{A}$ and ϕ are the vector and scalar potential. This Hamiltonian generates a mapping M which relates a particle's coordinates $\bar{x}\ \bar{y}\ \bar{t}\ \bar{p}_x\ \bar{p}_y\ \bar{p}_t$ as it leaves a beam line element to its coordinates $x\ y\ t\ p_x\ p_y\ p_t$ as it enters the element. See Fig. 1.

x y t
p_x p_y p_t
$\bar{x}$ $\bar{y}$ $\bar{t}$
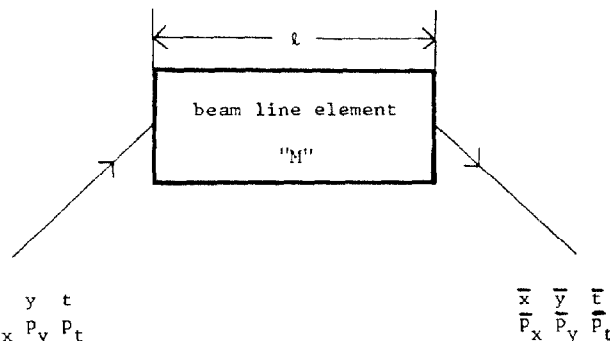$\bar{p}_x$ $\bar{p}_y$ $\bar{p}_t$

Fig. 1 The effect of a beam line element is to map a particle's coordinates x,y,t,p_x,p_y,p_t upon entry into their values $\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t$ as the particle leaves the element. This is described in terms of the action of a mapping M upon the coordinates.

Because the equations of motion for the trajectory with z as an independent variable are derived from a Hamiltonian, the mapping M from $\{x\ y\ t; p_x\ p_y\ p_t\}$ to $\{\bar{x}\ \bar{y}\ \bar{t}; \bar{p}_x\ \bar{p}_y\ \bar{p}_t\}$ is symplectic.

To proceed further, it is useful to replace the pair $(t,\ p_t)$ by new variables (T, P_T) representing deviations from the design arrival time at position z, $t^0(z)$, and design energy, $-p_t^0$. Such a transformation is given by

$$t = T + t^0(z)$$
$$p_t = P_T + p_t^0 \qquad (1)$$

When this is done, all variables vanish on the design orbit, and are small for nearby orbits. The transformation (1) is canonical, and leads to the new Hamiltonian

$$H(x\ y\ T; p_x\ p_y\ P_T) = -c^{-1}\{(p_t^0 + P_T + q\phi)^2 - m_0^2 c^4$$
$$-c^2(p_x - qA_x)^2 - c^2(p_y - qA_y)^2\}^{1/2} \qquad (2)$$
$$-qA_z - v_0^{-1}(P_T + p_t^0)$$

where v_0 is the design velocity.

Corresponding to H we define a Lie operator :H: by the rule

$$:H:g = [H,g] \qquad (3)$$

where g is any function and [,] denotes the Poisson bracket operation. It follows that the mapping M is given by the formal expression[4]

$$M = \exp(-\ell :H:). \qquad (4)$$

In general H has a power series expansion of the form

$$H = H_2 + H_3 + H_4 + \cdots \qquad (5)$$

where H_n is a homogeneous polynomial of degree n. Consequently M can be factored into an expression of the form[5]

$$M = \exp(-\ell:G_2:)\exp(-\ell:G_3:)\exp(-\ell:G_4:)\cdots \qquad (6)$$

Here the G_n are also homogeneous polynomials of degree n. In particular, the Campbell-Baker-Hausdorff (CBH) formula gives the explicit expressions

$$G_2 = H_2 \ , \quad G_3 = \{[1-\exp(\ell:G_2:)]/(-\ell:G_2:)\}H_3 \ , \quad \cdots \qquad (7)$$

In the derivation of (6) it has been assumed that the Hamiltonian (2) is independent of z. However, it can be shown that an expression of the form (6) holds in the general case.

Sample Calculations

Consider a quadrupole of length ℓ. Then

$$A_z = a_2(x^2 - y^2) \qquad (8)$$

and

$$H_2 = \frac{1}{2}\frac{m_0^2 c^4}{p_0^3 c^4}P_T^2 + \frac{1}{2p_0}(p_x^2 + p_y^2) - qa_2(x^2 - y^2) \qquad (9)$$

$$H_3 = -\frac{1}{2}\frac{p_t^0 m_0^2 c^4}{p_0^5 c^6}P_T^3 - \frac{1}{2}\frac{p_t^0}{p_0^3 c^2}P_T(p_x^2 + p_y^2)$$

Here p_0 is the design momentum. Upon evaluating (7), one finds

$$G_3 = -\frac{p_t^0 k}{8p_0 c^2}\{x^2 P_T[2\ell k - \sinh 2\ell k]$$

$$- \frac{2xp_x P_T}{kp_0}[1 - \cosh 2\ell k] - \frac{p_x^2 P_T}{k^2 p_0^2}[2\ell k + \sinh 2\ell k]$$

$$- y^2 P_T[2\ell k - \sin 2\ell k] - \frac{2yp_y P_T}{kp_0}[1 - \cos 2\ell k]$$

$$- \frac{p_y^2 P_T}{k^2 p_0^2}[2\ell k + \sin 2\ell k]\} + \frac{p_t^0 m_0^2 c^4}{2p_0^5 c^6}P_T^3 \qquad (10)$$

where $k \equiv \sqrt{2qa_2/p_0}$

Next consider a sextupole of length ℓ. Then

$$A_z = a_3(x^3 - 3xy^3) \qquad (11)$$

and

$$H_3 = -\frac{1}{2}\frac{p_t^0 m_0^2 c^4}{p_0^5 c^6}P_T^3 - \frac{1}{2}\frac{p_t^0}{p_0^3 c^2}P_T(p_x^2 + p_y^2)$$

$$- qa_3(x^3 - 3xy^2) \qquad (12)$$

$$H_2 = \frac{1}{2}\frac{m_0^2 c^4}{p_0^3 c^4}P_T^2 + \frac{1}{2p_0}(p_x^2 + p_y^2)$$

upon evaluating (7) one finds

$$G_3 = \frac{p_t^0}{2p_0^3 c^2}P_T(p_x^2 + p_t^2) + \frac{p_t^0 m_0^2 c^4}{2p_0^5 c^6}P_T^3 \qquad (13)$$

$$+ \ell qa_3(x^3 - 3xy^2) - \frac{3\ell^2 qa_3}{2p_0}(x^2 p_x - p_x y^2 - 2xyp_y)$$

$$- \frac{\ell^3 qa_3}{p_0}(xp_y^2 - xp_y^2 + 2yp_x p_y) - \frac{\ell^4 qa_3}{4p_0^3}(p_x^3 - 3p_x p_y^2)$$

The short sextuple approximation may be obtained by taking the limit $\ell \to 0$ and $\ell H_3 + F_3 = a_3(x^3 - 3xy^3)$. In this case, one gets

$$M = \exp:F_3: \qquad (14)$$

Similarly, the expression for a drift of length ℓ may be derived either from (8) and (10) or (12) and (13) by setting $a_2 = 0$ or $a_3 = 0$. Therefore for a drift of length ℓ, G_2 is as in (7) and (12) and

$$G_3 = -\frac{p_t^0}{2p_0^2 c^2}P_T(p_x^2 + p_y^2) - \frac{p_t^0 m_0^2 c^4}{2p_0^5 c^6}P_T^2 \qquad (15)$$

For a normal-entry dipole bend, it is most natural to work in cylindrical coordinates. Let ρ, ϕ lie in the midplane, and let z measure displacements out of the midplane. The methods outlined above give the results

$$M = \exp(-\phi_0 :G_2:)\exp(-\phi_0 :G_3:) \qquad (16)$$

with

$$G_2 = \frac{\rho_0}{2p_0}(p_r^2 + p_z^2) + \frac{\rho_0 m_0^2 c^4}{2p_0^3 c^4}P_T^2$$

$$- \frac{p_t^0}{p_0 c^2}r P_T + \frac{1}{2}\frac{\rho_0}{p_0}r^2, \qquad (17)$$

$$G_3 = \frac{p_z^2}{2p_0}\left[-(r - \frac{p_t^0 \rho_0}{p_0 c^2}P_T)\sin\phi_0 + (1 - \cos\phi_0)\frac{\rho_0 p_r}{p_0}\right]$$

$$+ \frac{m_0^2 c^4}{2p_0^3 c^4}P_T^2\left[-(r - \frac{p_t^0 \rho_0}{p_0 c^2}P_T)\sin\phi_0 + (1 - \cos\phi_0)\frac{\rho_0 p_r}{p_0}\right]$$

$$- \frac{\rho_0}{6p_0^2}\left[(r - \frac{p_t^0 \rho_0}{p_0 c^2}P_T)\sin\phi_0 + \frac{\rho_0 p_r}{p_0}\cos\phi_0\right]^3 + \frac{\rho_0 p_r^3}{6p_0^2}$$

Here ρ_0 and ϕ_0 are the design bend radius and angle of bend, and $r = \rho - \rho_0$ is the deviation of the trajectory from the design radius. The expressions for other common beam elements, e.g. bunchers and solenoids, can be worked out in a similar manner.

Comparison to Matrix Methods

The first constraint placed on the method outlined above is that it duplicate the results obtained by traditional matrix methods. We have chosen to benchmark our results against those employed in the computer code TRANSPORT. The transfer maps obtained by the above method for quadrupole, sextupole, drift and normal entry dipole all give results equivalent to those obtained using the matrix theory on which TRANSPORT is based[6]. This is reassuring, as the validity of TRANSPORT has been confirmed by its widespread success in the design of beam lines and spectrometers. For example, one finds for a sextupole the result

$$\bar{p}_x = M p_x = \exp(-\ell : G_2 :)\ \exp(-\ell : G_3 :)p_x$$

$$= p_x - \ell k^2 p_0 (x^2 - y^2) - \ell^2 k^2 (x p_x - y p_y) \quad (18)$$

$$- \frac{\ell^3 k^2}{3 p_0}(p_x^2 - p_y^2)$$

Lie algebraic methods are similar to matrix methods in that an entire beam line may be simulated by concatenation of mappings for the individual components. Instead of products of matrices, however, one combines the exponents of Lie transformations using the CBH theorem. The result is a single Lie transformation which specifies the trajectory of a particle through the entire beam line. Such mappings may be utilized in a variety of ways beyond simple transport, such as studies of resonances[2] or effects of beam-beam interactions[7].

Conclusions

A method for deriving transfer maps through beam-line elements has been given, illustrative calculations presented, and a brief catalogue of results for a variety of elements provided. Transfer map methods may be applied in much the same manner as matrix theories of beam transport. Moreover, Lie algebraic methods provide well-defined procedures for working to arbitrarily high order. One need only make the expansion (5) and work out the factorization (6). These methods therefore appear to provide a basis for the construction of efficient numerical codes for the computation of charged particle beam transport.

Notes and References

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