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# DUALITY THEOREM OF NONSTATISTICAL AND STATISTICAL BEAM PHASE SPACES

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## Summary

A duality theorem between nonstatistical and statistical beam phase spaces is found for effecting the translation of all the formulae and results of beam transport system in terms of nonstatistical phase space directly into the corrosponding ones in terms of statistical phase space and vice versa.

## Introduction

The basic goal of a beam transport system design is to get the required phase space at any point ( x, y, z, t) with given transport elements and initial phase space. There are two approaches to tackle the task, i.e. the nonstatistical phase space and statistical phase space. The beam transport theory in terms of nonstatistical phase space [1] is more intensively developed than that in terms of statistical phase space. For example, it was not until 1972 that C.R.Emigh first proved the statistical two-dimensional phase space conservation upon linear force assumption [2].

It turns out that the nonstatistical phase space and the statistical phase space satisfy the same differential equation. Consequently, we get the duality theorem between nonstatistical phase space and statistical phase space, by means of which all the formulae and results of beam transport system in terms of nonstatistical phase space can be translated directly into the corresponding ones in terms of statistical phase space and vice versa.

## Transport of a single particle

The motion of a single particle in beam transport system obeys Newton's second law:

$$\frac{dP_x}{dt}=F_x(X,Y,Z,P_x,P_y,P_z,t)$$
$$\frac{dP_y}{dt}=F_y(X,Y,Z,P_x,P_y,P_z,t)$$
$$\frac{dP_z}{dt}=F_z(X,Y,Z,P_x,P_y,P_z,t)$$

with relativistical relation between velocity and momentum

$$\frac{dX}{dt}=\frac{cP_x}{[P_x^2+P_y^2+P_z^2+m^2c^2]^{\frac12}}$$
$$\frac{dY}{dt}=\frac{cP_y}{[P_x^2+P_y^2+P_z^2+m^2c^2]^{\frac12}}$$
$$\frac{dZ}{dt}=\frac{cP_z}{[P_x^2+P_y^2+P_z^2+m^2c^2]^{\frac12}}$$

By Taylor expansion around the central trajectory $X_0(t)$, $P_{x0}(t)$, $Y_0(t)$, $P_{y0}(t)$, $Z_0(t)$, $P_{z0}(t)$ and keeping first-order terms $x=X-X_0$ , $p_x=P_x-P_{x_0}$ , $y=Y-Y_0$ , $p_{y_0}=P_y-P_{y_0}$ , $z=Z-Z_0$ , $p_z=P_z-P_{z_0}$ , we get the equation of motion:

$$\begin{aligned}
\frac{dx}{dt}&=0+b_{12}P_x+0+b_{14}P_y+0+b_{16}P_z\\
\frac{dP_x}{dt}&=b_{21}x+b_{22}P_x+b_{23}y+b_{24}P_y+b_{25}z+b_{26}P_z\\
\frac{dy}{dt}&=0+b_{32}P_x+0+b_{34}P_y+0+b_{36}P_z\\
\frac{dP_y}{dt}&=b_{41}x+b_{42}P_x+b_{43}y+b_{44}P_y+b_{45}z+b_{46}P_z\\
\frac{dz}{dt}&=0+b_{52}P_x+0+b_{54}P_y+0+b_{56}P_z\\
\frac{dP_z}{dt}&=b_{61}x+b_{62}P_x+b_{63}y+b_{64}P_y+b_{65}z+b_{66}P_z
\end{aligned} \tag{1}$$

where $b_{ij}$ are functions of time t.

With the 6-dimensional phase space vector $V=(x,p_x,y,p_y,z,p_z)$ and transport system matrix $B=\begin{bmatrix} b_{11}\cdots\cdots b_{16}\\ \cdots\cdots\cdots\\ b_{61}\cdots\cdots b_{66}\end{bmatrix}$ , mathematically describing the transport elements, the equation of motion (1) can be put in a compact form:

$$\frac{dV}{dt}=B(t)V \tag{2}$$

Let $V^{(i)}(i=1,2,\ldots,6)$ be any six linearly independent solutions of differential equation (2) and define the principal solution matrix

$$T(t)=\begin{bmatrix} x^{(1)}(t) & \cdots x^{(6)}(t)\\ p_x^{(1)}(t) & \cdots P_x^{(6)}(t)\\ \cdots & \cdots\\ P_z^{(1)}(t) & \cdots P_z^{(6)}(t)\end{bmatrix}$$

which, by definition, satisfies the differential equation (2):

$$\frac{dT}{dt}=B(t)T \tag{3}$$

Differentiation of the determinant of T(t) gives

$$\frac{d}{dt}|T(t)|=\begin{vmatrix} \frac{dx^{(1)}}{dt} & \cdots & \frac{dx^{(6)}}{dt}\\ P_x^{(1)} & \cdots & P_x^{(6)}\\ y^{(1)} & \cdots & y^{(6)}\\ P_y^{(1)} & \cdots & P_y^{(6)}\\ z^{(1)} & \cdots & z^{(6)}\\ P_z^{(1)} & \cdots & P_z^{(6)}\end{vmatrix}+\cdots+\begin{vmatrix} x^{(1)} & \cdots & x^{(6)}\\ P_x^{(1)} & \cdots & P_x^{(6)}\\ y^{(1)} & \cdots & y^{(6)}\\ P_y^{(1)} & \cdots & P_y^{(6)}\\ z^{(1)} & \cdots & z^{(6)}\\ \frac{dP_z^{(1)}}{dt} & \cdots & \frac{dP_z^{(6)}}{dt}\end{vmatrix}$$

$$=(b_{11}+b_{22}+\cdots+b_{66})|T(t)|$$

and integration back yields:

$$|T(t)|=|T(t_0)|\,e^{\int_{t_0}^{t}(b_{11}+b_{22}+\cdots+b_{66})dt} \tag{4}$$



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The general solution of the differential equation (2) is found to be $V(t)=T(t)C$, where the constant vector C is defined by the initial condition $V(t_0)=T(t_0)C$, giving the final transport equation of a single particle:

$$V(t)=T(t)T^{-1}(t_0)V(t_0)=R(t)V(t_0) \qquad (5)$$

with transport matrix $R(t)=T(t)T^{-1}(t_0)$.

By (4), we get the determinant of the transport matrix

$$|R(t)|=|T(t)T^{-1}(t_0)|=e^{\int_{t_0}^{t}(b_{11}+b_{22}+\cdots+b_{66})dt} \qquad (6)$$

Differentiation of (5) gives the differential equation of transport matrix:

$$\frac{dR}{dt}=\frac{dV}{dt}V_0^{-1}=BVV_0^{-1}=BRV_0V_0^{-1}=BR \qquad (7)$$

## Transport of a particle beam

## — nonstatistical case

For the nonstatistical case, a particle beam is analogized by a 6-dimensional phase space ellipsoid $\tilde{V}\sigma^{-1}V=1$, where the mathematical description of the phase space is represented by the matrix $\sigma=\begin{bmatrix}\sigma_{11}\cdots\sigma_{16}\\ \cdots\cdots\\ \sigma_{61}\cdots\sigma_{66}\end{bmatrix}$

with its volume $\frac{\pi^3}{6}|\sigma|$.

Let the initial phase space be $\tilde{V}_0\sigma_0^{-1}V_0=1$ and, by substitution of (5), get

$\tilde{V}\tilde{R}^{-1}\sigma_0^{-1}R^{-1}V=\tilde{V}\sigma^{-1}V=1$ and, hence, the transport equation of phase space

$$\sigma(t)=R(t)\sigma_0\tilde{R}(t) \qquad (8)$$

Combination of (6) and (8) gives the transport equation of the phase space volume:

$$|\sigma(t)|=|\sigma_0|e^{2\int_{t_0}^{t}(b_{11}+b_{22}+\cdots+b_{66})dt}, \qquad (9)$$

showing that if $\int_{t_0}^{t}(b_{11}+b_{22}+\cdots+b_{66})dt=0$, the phase space volume is conserved.

Remenbering $b_{11}=b_{33}=b_{55}=0$ by (1), if $b_{22}=b_{44}=b_{66}=0$, i.e. if the force along any direction is independent of the momentum along that direction, the phase space volume is conserved.

Furthermore, if any of $b_{22}$, $b_{44}$, $b_{66}$ is nonzero, but $b_{22}+b_{44}+b_{66}=0$, i.e. the divergence of force is zero $\left(\nabla_p\cdot\dot{P}=\frac{\partial\dot{P}_x}{\partial P_x}+\frac{\partial\dot{P}_y}{\partial P_y}+\frac{\partial\dot{P}_z}{\partial P_z}=0\right)$ the phase space volume is conserved.

Finally, if the divergence of force is nonzero, but its integral is zero $\left(\int_{t_0}^{t}\nabla_p\cdot\dot{P}dt=0\right)$ the phase space volume is conserved.

All these three cases give more relaxed condition than the conservative system requirement by Liouville's theorem for phase space conservation.

With account of (7), differentiation of (8) gives

$$\frac{d\sigma}{dt}=\frac{dR}{dt}\sigma_0\tilde{R}+R\sigma_0\frac{d\tilde{R}}{dt}=BR\sigma_0\tilde{R}+R\sigma_0\tilde{R}\tilde{B}$$

Substituting (8) in it again, we get the differential equation satisfied by the nonstatistical phase space $\sigma$

$$\frac{d\sigma}{dt}=B\sigma+\widetilde{B\sigma} \qquad (10)$$

For any given initial nonstatistical phase space $\sigma_0$ and any given transport elements or their mathematical description $B(t)$, solution of the differential equation (10) yields nonstatistical phase space $\sigma$ at any position and any time, thus fulfilling the task of the beam transport system.

## Tramsport of a particle beam

## — statistical case

For the statistical case, a particle beam is described by a statistical 6-dimensional phase space

$$\langle V\tilde{V}\rangle=\begin{bmatrix}\langle x_1^2\rangle & \langle x_1x_2\rangle & \cdots & \langle x_1x_6\rangle\\ \langle x_2x_1\rangle & \langle x_2^2\rangle & \cdots & \langle x_2x_6\rangle\\ \cdot & \cdot & \cdots & \cdot\\ \langle x_6x_1\rangle & \langle x_6x_2\rangle & \cdots & \langle x_6^2\rangle\end{bmatrix} \qquad (11)$$

where $\langle x_ix_j\rangle$ is the averaged value of $x_ix_j$ over the whole phase space $\Omega$ with any distribution $\psi$ i.e.

$$\langle x_ix_j\rangle=\int x_ix_j\psi d\Omega=\int x_ix_jdN=\int x_ix_j\psi_0d\Omega_0 \qquad (12)$$

upon the assumption of particle conservation $dN=\psi d\Omega=\psi_0d\Omega_0$.

By double use of (5) we get $V\tilde{V}=R(t)V_0\tilde{V}_0\tilde{R}(t)$ and averaging it over the whole phase space with any distribution, find the transport equation of the statistical phase space:

$$\langle V\tilde{V}\rangle=R(t)\langle V_0\tilde{V}_0\rangle\tilde{R}(t) \qquad (13)$$

which is the statistical counterpart of the nonstatistical equation (8).

With account of (7), differentiation of (13) gives

$$\frac{d}{dt}\langle V\tilde{V}\rangle=\frac{dR}{dt}\langle V_0\tilde{V}_0\rangle\tilde{R}+R\langle V_0\tilde{V}_0\rangle\frac{d\tilde{R}}{dt}=BR\langle V_0\tilde{V}_0\rangle\tilde{R}+R\langle V_0\tilde{V}_0\rangle\tilde{R}\tilde{B}$$

Substituting (13) in it again, we get the differential equation* satisfied by the statistical phase space $\langle V\tilde{V}\rangle$

$$\frac{d}{dt}\langle V\tilde{V}\rangle=B\langle V\tilde{V}\rangle+\widetilde{B\langle V\tilde{V}\rangle} \qquad (14)$$

which is the statistical counterpart of the nonstatistical equation (10).

For any given initial statistical phase space $\langle V_0\tilde{V}_0\rangle$ and any given transport elements or their mathematical description $B(t)$, solution of the differential equation (14) yields statistical phase space $\langle V\tilde{V}\rangle$ at any position and any time, thus fulfilling the task of the beam transport system.

*F.J.Sacherer got a similar equation for a special case. Cf. IEEE, NS-18 1105 (1971).

## Duality theorem of nonstatistical and statistical phase spaces

Equations (10) and (14) show that the nonstatistical phase space $\sigma$ and statistical phase space $\langle V\tilde{V}\rangle$ satisfy the same differential equation. By the existence theorem of differential equation, if the initial conditions differ with a constant k, i.e. $\sigma_0 = k\langle V_0\tilde{V}_0\rangle$, so do the solutions, i.e. $\sigma(t) = k\langle V(t)\tilde{V}(t)\rangle$, thus giving

Duality theorem of nonstatistical and statistical phase spaces:

If $\sigma_0 = k\langle V_0\tilde{V}_0\rangle$,

then $\sigma(t) = k\langle V(t)\tilde{V}(t)\rangle$ or $\sigma_{ij} = k\langle x_i(t)x_j(t)\rangle$ (15)

By means of the duality theorem we can translate all the formulae and results of the beam transport system in terms of nonstatistical phase space $\sigma$ direclty into the corresponding ones in terms of statistical phase space $\langle V\tilde{V}\rangle$. For instance, substitution of (15) in (9) with cancellation of constant k gives

$$|\langle V(t)\tilde{V}(t)\rangle| = |\langle V_0\tilde{V}_0\rangle|\, e^{2\int_{t_0}^{t}(b_{11}+b_{22}+\cdots+b_{66})dt} \qquad (16)$$

which is the statistical counterpart of the nonstatistical equation (9).

In the following is given a list of some important dual formulae of nonstatistical and statistical phase spaces:

| Nonstatistical phase space $\sigma$ | Statistical phase space $\langle V\tilde{V}\rangle$ |
|---|---|
| 1. Phase space transport $\sigma(t) = R(t)\sigma_0\tilde{R}(t)$ | $\langle V(t)\tilde{V}(t)\rangle = R(t)\langle V_0\tilde{V}_0\rangle\tilde{R}(t)$ |
| 2. Phase volume transport $\lvert\sigma(t)\rvert = \lvert\sigma_0\rvert\, e^{2\int_{t_0}^{t}(b_{11}+b_{22}+\cdots+b_{66})dt}$ | $\lvert\langle V(t)\tilde{V}(t)\rangle\rvert = \lvert\langle V_0\tilde{V}_0\rangle\rvert\, e^{2\int_{t_0}^{t}(b_{11}+b_{22}+\cdots+b_{66})dt}$ |
| 3. Phase volume of projections on 2-dimensional subspace $(x_i, x_j)$ $\sqrt{\begin{vmatrix}\sigma_{ii} & \sigma_{ij}\\ \sigma_{ji} & \sigma_{jj}\end{vmatrix}}$ | $\sqrt{\begin{vmatrix}\langle x_i^2\rangle & \langle x_ix_j\rangle\\ \langle x_jx_i\rangle & \langle x_j^2\rangle\end{vmatrix}}$ |
| 4. Relations between the elements, assuming $x_j = x_i'$, where i=1,3,5 with corresponding j=2,4,6 respectively $\sigma_{ii}' = 2\sigma_{ij}$ | $\langle x_i^2\rangle' = 2\langle x_ix_j\rangle$ |
| 5. Beam envelopes $\sigma_{ii}$ | $\langle x_i^2\rangle$ |
| 6. Conditions of waist, assuming $x_j = x_i'$, where i=1,3,5 with corresponding j=2,4,6 respectively $\sigma_{ij} = 0$ | $\langle x_ix_j\rangle = 0$ |
| 7. Beam envelope equations, assuming $x_j = x_i'$, where i=1,3,5 with corresponding j=2,4,6 respectively $\frac{d^2\sqrt{\sigma_{ii}}}{dt^2} - \frac{\begin{vmatrix}\sigma_{ii} & \sigma_{ij}\\ \sigma_{ji} & \sigma_{jj}\end{vmatrix}}{(\sqrt{\sigma_{ii}})^3} - \frac{\sigma_{ij}' - \sigma_{jj}}{\sqrt{\sigma_{ii}}} = 0$ | $\frac{d^2\sqrt{\langle x_i^2\rangle}}{dt^2} - \frac{\begin{vmatrix}\langle x_i^2\rangle & \langle x_ix_j\rangle\\ \langle x_jx_i\rangle & \langle x_j^2\rangle\end{vmatrix}}{(\sqrt{\langle x_i^2\rangle})^3} - \frac{\langle x_ix_j\rangle' - \langle x_j^2\rangle}{\sqrt{\langle x_i^2\rangle}} = 0$ |
| 8. Waist to waist transport, assuming $x_j = x_i'$, where i=1,3,5 with corresponding j=2,4,6 respectively $R(t) = \begin{bmatrix}\sqrt{\frac{\beta_2}{\beta_1}}\cos\phi & \sqrt{\beta_1\beta_2}\sin\phi\\ \frac{-1}{\sqrt{\beta_1\beta_2}}\sin\phi & \sqrt{\frac{\beta_1}{\beta_2}}\cos\phi\end{bmatrix}$, $\beta_{1,2} = \left[\frac{\sigma_{ii}}{\sigma_{jj}}\right]_{1,2}^{\frac{1}{2}}$ where $\beta_1$ and $\beta_2$ are characteristic length at the first and second waist respectively. | $R(t) = \begin{bmatrix}\sqrt{\frac{\beta_2}{\beta_1}}\cos\phi & \sqrt{\beta_1\beta_2}\sin\phi\\ \frac{-1}{\sqrt{\beta_1\beta_2}}\sin\phi & \sqrt{\frac{\beta_1}{\beta_2}}\cos\phi\end{bmatrix}$, $\beta_{1,2} = \left[\frac{\langle x_i^2\rangle}{\langle x_j^2\rangle}\right]_{1,2}^{\frac{1}{2}}$ |


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