IEEE Transactions on Nuclear Science, Vol. NS-28, No. 3, June 1981

ACCELERATING PARTICLES WITH TRANSVERSE CAVITY MODES

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Abstract: Calculations on particle acceleration with laser-like modes in a spheroidal cavity show that the energy transfer is acceptable only when the normal wall fields are appreciable. The transverse modes with potentially large electric field strengths can not be used for electron acceleration.

High electric fields are obviously advantageous in powerful electron accelerators: for a given energy gain per particle, the device length is inversely proportional to the maximum field; and the maximum current is roughly proportional to the field.

We have proposed[1] a laser-inspired accelerating method termed the transverse RF accelerator; This concept tries to use laser-like electric fields that are ultimately limited by skin current heating: the peak electric field could then be up to 1 GeV/m. The transverse RF accelerator tries to get a high strength by using an accelerator cavity analogous to an optical resonator: The focusing of electromagnetic energy produces large fields along the cavity axis for manageable fields at the wall. This paper discusses the field structure as related to the acceleration mechanism, and reports on preliminary numerical results. Important aspects that are not discussed here are the accelerator power requirements, the quality factor Q, the shunt impedance, and other cavity parameters. These can, in principle, be evaluated from the exact fields, but this has yet to be done.

The accelerator cavity can be conceptually generated from a linear optical resonator consisting of two hollow mirrors. The resonator is then rotated about an axis in the focal plane and extended axially to allow wavelengths comparable to cavity dimensions. The resulting cavity is the ellipsoid, shown in Fig. 1: The cavity exhibits radial focusing, from the cylindrical symmetry, but in addition there is axial focussing due to the curvature in the cavity walls.

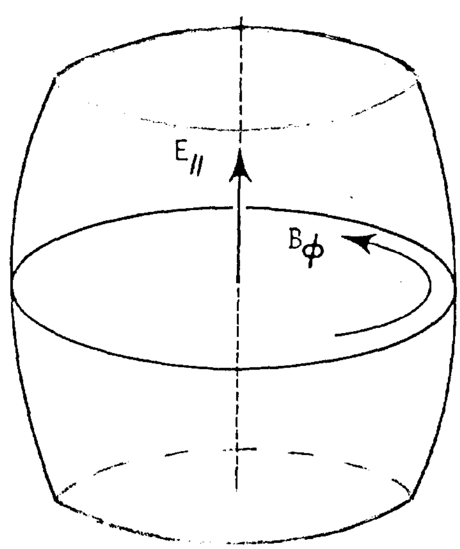

Figure 1
The transverse RF accelerator cavity with the major electromagnetic field components.

The novel points in the proposed transverse RF accelerator are threefold: i. use of laser-like higher order cavity modes in approximate resonance with relativistic electrons, ii. electric fields at the wall that are largely transverse, and iii. the accelerating electric fields on axis are enhanced by radial focusing.

The numerical evaluations of the energy transfer performed up to now corroborate previous analytical[1] results only in part. Most disturbing is that the energy transfer from the exact cavity fields is lower than previously estimated.

The preliminary conclusion for our analyses is that the ultimate performance of such a cavity is measured by the ratio $\Delta E/eE_\perp L$, and that this ratio is limited by a number of order unity. Here ΔE is the energy transfer to the electrons, L the cavity length, and $E_\perp$ a typical normal wall field. This value is comparable to that obtained in conventional pill-box cavities; However, our cavity could have a factor of order unity advantage over conventional cavities because the fields here are focused axially: more work is needed to verify this statement.

Although we know of no fundamental limitations indicating that the above ratio is a fundamental limit for all cavities, it appears for the moment that the decisive way to improve the ultimate accelerating gradient is to increase the breakdown limit for $E_\perp$, e.g. by using more suitable wall materials or local magnetic insulation.

The electromagnetic fields envisioned for our ellipsoidal cavity have the electric vector mainly parallel to the cavity axis, while the magnetic field is roughly perpendicular to the axis. These conditions are satisfied by one case that is exactly solvable.[2] In this particular case all fields can be derived from one magnetic field component $\underline{B}_\phi$ (see Figure 1).

The free oscillations of the ellipsoidal resonator are solutions of

$$\nabla \times \nabla \times \underline{B} = k^2 \underline{B} \tag{1}$$

A harmonic time dependence exp $i\omega t$ is assumed, $k = \omega/c$; the boundary condition is that the electric field parallel to the cavity wall be zero at the wall.

The vector wave equation can be simplified to a scalar wave equation when the magnetic field is purely azimuthal, with the relation[3]

$$-\nabla \times \nabla \times \left[\underline{e}_\phi \psi\right] = \underline{e}_\phi \left[\nabla^2 \psi - \frac{\psi}{r^2}\right], \tag{2}$$

where $\underline{e}_\phi$ is the unit vector in the azimuthal direction, and $\psi = \psi(r,z)$ is rotationally symmetric. The $-\psi/r^2$ term can be conceptually generated from the azimuthal mode $\psi \cos \phi$ (m=1) in the Laplacian; the lowest azimuthal mode in the vector wave equation corresponds to the first mode in the scalar equation.

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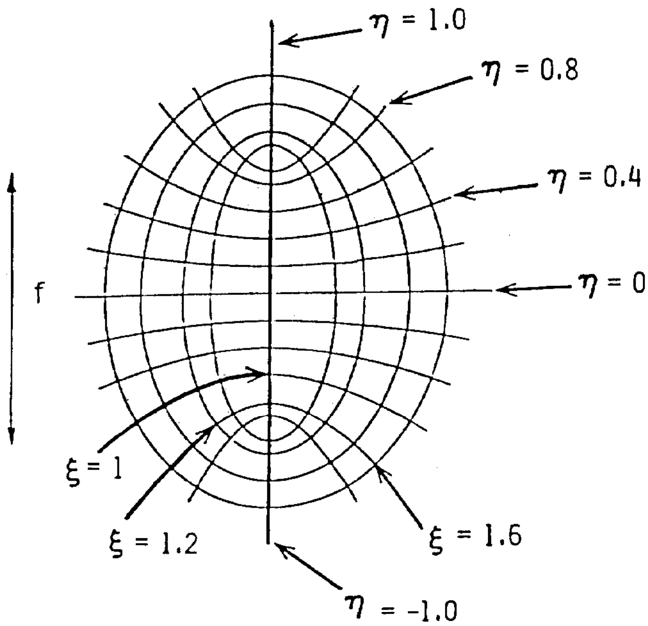

Figure 2
The spheroidal coordinate system.

The prolate ellipsoidal coordinate system is appropriate to the boundary conditions. The coordinates are defined as in Figure 2,

$$x = r\cos\psi, \quad y = r\sin\psi \quad , \tag{3}$$

$$r = \frac{f}{2}\,(1-\eta^2)^{\frac{1}{2}}\,(\xi^2-1)^{\frac{1}{2}} \quad ,$$

$$z = \frac{f}{2}\,\eta\,\xi \quad ,$$

where f is the focal distance.

Equation (2) is separable in spheroidal coordinates, and the exact magnetic field is given as a product of two spheroidal wave functions

$$B_\phi(\xi,\eta) = \text{const.}\; R_{1n}(h,\xi)\; S_{1n}(h,\eta) \quad , \tag{4}$$

where n is the axial mode number, and h is the parameter

$$h = \frac{kf}{2} \quad .$$

The radial spheroidal wave function $R_{1n}(h,\xi)$ and and the axial function $S_{1n}(h,\eta)$ satisfy the same differential equation

$$\left[\frac{d}{dz}\,(1-z^2)\,\frac{d}{dz} + \lambda_{1n} - h^2z^2 - \frac{1}{1-z^2}\right] u = 0 \tag{5}$$

but in two different regions, $|\eta| \leq 1$ and $1 \leq \xi < \infty$.

The electric field component parallel to the cavity wall $\xi = \xi_0 =$ constant, is (from $\nabla \times \underline{B}$):

$$E_\eta = -\frac{i}{ch}\,\frac{1}{\sqrt{\xi^2-\eta^2}}\,\frac{\partial}{\partial\xi}\sqrt{\xi^2-1}\;B_\phi \quad ; \tag{6a}$$

and the normal field is

$$E_\xi = \frac{i}{ch}\,\frac{1}{\sqrt{\xi^2-\eta^2}}\,\frac{\partial}{\partial\eta}\sqrt{1-\eta^2}\;B_\phi \quad . \tag{6b}$$

The boundary condition that determines the resonance frequency[4] is

$$E_\eta = 0 = \frac{\partial}{\partial\xi}\left[\sqrt{\xi^2-1}\;\; R_{1n}(h,\xi)\right] = 0 \quad . \tag{6c}$$

An acceptable approximation to the frequency can be obtained with the approximation[5]

$$\sqrt{\xi^2-1}\;\; R_{1n}(h,\xi) \simeq \left(\frac{\pi}{2h}\right)^{\frac{1}{2}}\left(\frac{s}{s'}\right)^{\frac{1}{2}} J_1(s) \quad , \tag{7a}$$

where J_1 is the Bessel function of first order,

$$s = s(h,n;\xi) \simeq h\sqrt{\xi^2-1} - \frac{q}{2}\,\text{arc cos}\;\xi^{-1} - \frac{q^2+3}{16h\xi^2}\sqrt{\xi^2-1} \quad , \tag{7b}$$

and $q = 2n-1$. The factor $(s/s')^{\frac{1}{2}}$ is a slowly varying function of ξ, that can be ignored. Then, approximately,

$$s = j'_{1,\ell} \quad ;$$

therefore, neglecting the last term in Equation (7b):

$$h = (\xi_0^2-1)^{-\frac{1}{2}}\left(j'_{1\ell} + \frac{2n-1}{2}\,\text{arc cos}\;\xi_0^{-1}\right) . \tag{8}$$

Here $\ell = 1$ corresponds to a half-wave cavity, and $\ell = 2$ to a 3/2-wave cavity.

The energy transfer from the fields to the accelerating electrons is calculated by a Fourier transform over the fields on axis: The exact electric field along the axis is

$$E_\eta = E_p\,N_n\,(1-\eta^2)^{-\frac{1}{2}}\,S_{1n}(h,\eta) \quad , \tag{9}$$

where E_p is the peak field value, and N_n is a normalization constant. Asymptotically, for h>>1, in a region of width $h^{-\frac{1}{2}}$ about zero[6] and with $r = n-1$

$$E_\eta = E_p\,N_n\,H_r(\eta\sqrt{h})\;\exp - h\,\eta^2/2 \quad . \tag{10}$$

Previous estimates[1] of energy transfer were based on the counterpart of Equation (10), for an approximately scalar version of Equation (1). The difference between the two calculations is small, namely the Bessel function zero j'_{1n} is replaced by j_{0n}.

Figure 3 compares the energy transfer computed from the Hermite-Gaussian approximation (10) to the exact result using Equation (9) for the case $\xi_0 = \sqrt{2}$ (a confocal cavity). Shown is the normalized energy transfer $\Delta E_\parallel / eE_p\lambda$ as function of axial mode number n, parametrized by ℓ. The surprise here is that the agreement is not uniformly good: what is worse, the values from the Hermite-Gaussian approximation are too high, by a large factor.

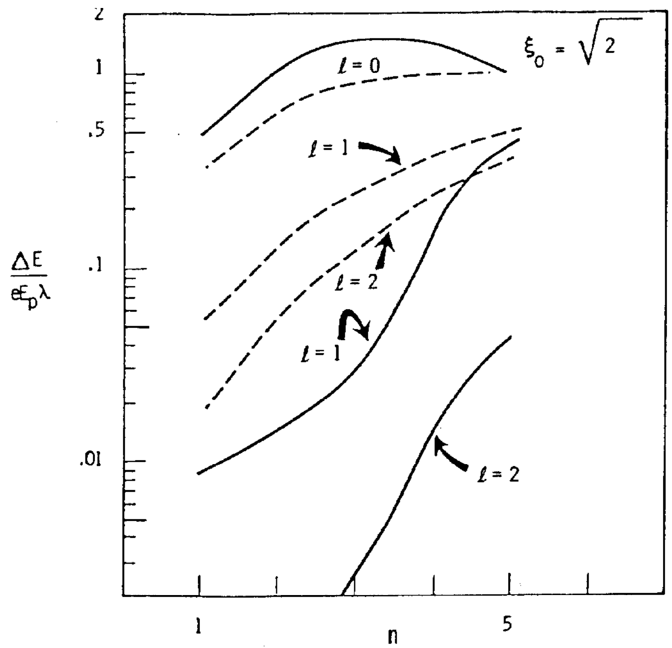

Figure 3.

Comparison between the Hermite-Gaussian analytical approximation (dashed line) and exact numerical data (solid line). n is the axial mode number.

The cause for this discrepancy seems to be that the Hermite-Gaussian approximation is not good in its "tails": The actual fields at the foci decrease more rapidly with increasing frequency h, as roughly as the sqaure of the Gaussian tails.

An exact formula for the energy transfer between the foci is obtained as follows:[7]

Multiply the differential Equation (5) for S_{1n} by

$$T \equiv e^{ih\eta} \Big/ \sqrt{1-\eta^2} \ , \tag{11}$$

and integrate from $\eta = -1$ to $\eta = 1$.

Integration by parts gives

$$\left[T'(1-\eta^2)S_{1n} - T(1-\eta^2)\, S_{1n}'\right]_{-1}^{1} = \tag{12}$$

$$\int_{-1}^{1} S_{1n} \left[\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta} + \lambda_{1n} - h^2\eta^2 - \frac{1}{1-\eta^2}\right] T \, d\eta \ .$$

The bracketed operator working on T simplifies considerably;

$$\left[\ \right] T = (1-\eta^2)^{-\frac{1}{2}} (\lambda_{1n}-h^2)\, e^{ih\eta} \ . \tag{13}$$

Therefore, the energy transfer integral U is

$$\int_{-1}^{1} \frac{S_{1n}(h,\eta)\, e^{ih\eta}}{\sqrt{1-\eta^2}}\, d\eta = \frac{\left[(1-\eta^2)(T'S-TS')\right]_{-1}^{1}}{\lambda_{1n}-h^2} . \tag{14}$$

The term in square brackets should be evaluated in the limit $|\eta| \to 1$; in this limit the function S_{1n} is

$$S_{1n}(h,\eta) \simeq N_{1n}(1-\eta^2)^{\frac{1}{2}}\, k_3 h/2 \ , \tag{15}$$

where k_3 for large h is found as[6]

$$k_3 = e^{-h}\, h^{n/2}\, 2^{(3n-1)/2} \sqrt{\pi} \ , \tag{16}$$

and the normalization factor N_{1n} is such that the maximum of S_{1n} is approximately unity:

$$N_{1n} \simeq \sqrt{(n-1)!\ \pi/2} \ . \tag{17}$$

The sqaure brackets in (14), exclusive of the sinusoidal term, then become simply $N_n\, k_3\, h$. The energy transfer integral becomes, approximately,

$$U \simeq \pi \sqrt{(n-1)!}\ h^{(n+s)/2}\, 2^{3n/2}\, e^{-h} \begin{matrix}\sin\\ \cos\end{matrix}\, h \,/(\lambda_{1n}-h^2) \ : \tag{18}$$

(the sin-function should be chosen for n even, cos-for n odd).

Equation (18) should be used with caution, because the estimate for the maximum value of S_{1n} contains an additional constant of order unity; however, the dominant behavior for large h is exp - h. This should be contrasted to the energy transfer[1] from the field as approximated by Equation (10), where the dominant behavior is exp - h/2.

The conclusion from these energy transfer calculations seems to be that higher order optical cavity modes couple to relativistic particles with a much lower efficiency than was previously thought. However, additional analysis and also numerical evaluations of the formulae involved are needed to corroborate this conclusion.

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