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EXACT NUMERICAL CALCULATION OF CHROMATICITY IN SMALL RINGS

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Abstract

A Newton's search method is presented which efficiently finds closed orbits in a ring whose lattice contains both linear elements, such as drifts and quadrupoles, and nonlinear elements such as dipoles with a small radius of curvature, sextupoles, etc. The method simultaneously determines the tune of the closed orbit. By observing how the location of the closed orbt depends on the total momentum, the η and η' functions are determined exactly (including nonlinear terms in δ). By observing how the tunes of the closed orbit depend on the total momentum, the chromaticities are determined exactly. The essential tool employed is the simultaneous integration of the equations of motion for a trajectory along with the variational equations for neighboring trajectories.

Introduction

Chromaticity correction is often essential for the operation of synchrotrons and storage rings. Methods for the calculation of chromaticity have been developed by several authors. They range from completely analytical calculations to hybrid calculations that make use of analytical results combined with numerical results from matrix lattice codes. In some cases, these different methods have been applied to the same problem with differing results. There is also concern that some methods omit nonlinear dipole contributions which can be very important for small rings. This paper describes a purely numerical method for chromaticity calculation that is both conceptually simple and exact. Its use can therefore serve as a benchmark for checking other methods.

Method of Computation

Briefly stated, the method of chromaticity calculation to be described is as follows:

1. Specify the machine lattice including dipole strengths, quadrupole strengths, sextupole strengths, etc.
2. Specify the momentum of a test particle.
3. Find the closed orbit corresponding to this momentum.
4. Find the tunes of this closed orbit.
5. Repeat steps 2 through 4 for a range of momentum values, observe how tunes vary with particle momentum, and thereby determine the chromaticity.

Obviously, the key elements in this procedure are steps 3 and 4. They are carried out with the aid of a numerical integration code which simultaneously integrates the equation of motion for a particle trajectory and the variational equations for neighboring trajectories.

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Variational Equations

Suppose the orbit equations for a lattice are written in the form

$$u'_i = f_i(\vec{u},\theta) \qquad i = 1....4. \tag{1}$$

Here $\vec{u}$ denotes a four-component vector whose entries are the two coordinates and two momenta transverse to the beam axis. The quantity θ denotes some generalized angle which increases by 2π in going around the lattice, and a prime denotes differentiation with respect to θ.

Variational equations describe all orbits near a particular orbit. Let $\vec{u}_*(\theta)$ denote a particular orbit of interest. Then orbits near this orbit can be written in the form

$$\vec{u} = \vec{u}_*(\theta) + \varepsilon\vec{w} \tag{2}$$

where ε is a small quantity. By definition, $\vec{u}_*(\theta)$ satisfies the equation (1). Inserting the prescription (2) into the equation of motion (1), and retaining terms of lowest order in ε, shows that $\vec{w}$ must satisfy the variational equation

$$\vec{w}' = A_*(\theta)\vec{w}. \tag{3}$$

Here A_* is a 4x4 theta dependent matrix defined by

$$A_{*ij}(\theta) = \partial f_i(\vec{u},\theta)/\partial u_j \Big|_{\vec{u} = \vec{u}_*(\theta)}. \tag{4}$$

Because the variational equations are linear, their solution for all initial conditions can be obtained by a finite amount of computation: Let $\theta = 0$ denote some arbitrary point in the lattice. Consider the first-order linear matrix variational equation defined by

$$B_*'(\theta) = A_*(\theta)\, B_*(\theta) \tag{5}$$

with the initial condition

$$B_*(0) = I. \tag{6}$$

Here B_* is a 4x4 matrix, I denotes the 4x4 identity matrix, and A_* is the same matrix as defined in (4). That is, the integration of (5) with the initial condition (6) is equivalent to the integration of 16 first-order equations. Now let $\vec{w}^0$ be an arbitrary four-component vector. Consider $\vec{w}(\theta)$ defined by the equation

$$\vec{w}(\theta) = B_*(\theta)\vec{w}^0. \tag{7}$$

It is easily checked that $w(\theta)$ is a solution to the variational equations (3) and satisfies the arbitrarily prescribed initial condition

$$\vec{w}(0) = \vec{w}^0. \tag{8}$$

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Determination of Closed Orbits

Consider an imaginary plane which intersects the design orbit at right angles somewhere in some straight section. Then it is obvious that every other orbit will also intersect this plane. Indeed, every orbit intersects this plane each time it goes around the lattice.

Furthermore, every orbit with a given fixed total momentum p is completely determined by the values of its two transverse coordinates and two transverse momenta at the moment of intersection. To see that this is the case, it is only necessary to realize that these four quantities may be viewed as a complete set of initial conditions for the four first-order orbit differential equations, and to recall that the solution of a set of differential equations is uniquely and completely specified by initial conditions. The whole situation may be summarized by saying that there is a certain 4-dimensional hypersurface in phase space which cuts across every phase space trajectory for the orbits under study. In addition, each orbit is uniquely specified in terms of any point (four coordinates) at which it crosses this surface. The 4-dimensional hypersurface just described is called a Poincare surface of section.

Next observe that orbits in the lattice generate a mapping M of the surface of section into itself. Consider a point on the surface of section. Since any such point requires four numbers for its specification, it is convenient to denote these four numbers collectively by a fourcomponent vector $\vec{a}$. Now use the coordinates of $\vec{a}$ as initial conditions, and follow the orbit with these initial conditions once around the lattice until it again crosses the surface of section at some point $\vec{b}$. The mapping M, called the Poincare map, is simply defined by the relation

$$\vec{b} = M\vec{a}. \tag{9}$$

That is, M describes the effect of one circuit around the lattice. Note that since the equations of motion are in general nonlinear, the relation between $\vec{b}$ and $\vec{a}$ defined by the Poincare map M is also nonlinear.

Much of what one wants to know about orbits is equivalent to a knowledge of M. For example, the determination of a closed orbit is equivalent to the discovery of a point $\vec{f}$, called a fixed point, which is sent into itself under the action of M,

$$M\vec{f} = \vec{f}. \tag{10}$$

That is, the set of initial conditions $\vec{f}$ for a closed orbit must, by definition, be mapped into itself by one circuit around the lattice.

Let $\vec{a}$ be an arbitrary point in the surface of section and let $\vec{a} + \vec{\varepsilon}$, where $\vec{\varepsilon}$ is a small vector, be a point near $\vec{a}$. Now consider the point $M(\vec{a} + \vec{\varepsilon})$. According to equation (9), this point should be near $\vec{b}$ since $\vec{a} + \vec{\varepsilon}$ is near $\vec{a}$. In fact, there is a power series expansion in $\vec{\varepsilon}$ of the form

$$M(\vec{a} + \vec{\varepsilon}) = \vec{b} + L_a\vec{\varepsilon} + O(\vec{\varepsilon}^2) \tag{11}$$

where L_a is a 4x4 matrix which will be called the linear part of M at a.

It is easy to see that the linear part of M is available from the variational equations. Suppose the surface of section is located at $\theta = 0$. Let $\vec{u}_a(\theta)$ be the orbit with initial conditions $\vec{a}$. That is,

$$\vec{u}_a(0) = \vec{a} \tag{12}$$

Then this trajectory must also satisfy the equation

$$\vec{u}_a(2\pi) = M\vec{a} = \vec{b}. \tag{13}$$

Next, with the aid of the variational equations, the nearby trajectory $\vec{u}_{a+\varepsilon}(\theta)$ with initial conditions $\vec{a} + \vec{\varepsilon}$ is expressible in the form

$$\vec{u}_{a+\varepsilon}(\theta) = \vec{u}_a(\theta) + B_a(\theta)\vec{\varepsilon} + O(\vec{\varepsilon}^2). \tag{14}$$

[See equations (2), (7), and (8).]

Now put $\theta = 2\pi$ in equation (14). The result is the relation

$$M(\vec{a} + \vec{\varepsilon}) = \vec{u}_{a+\varepsilon}(2\pi) = u_a(2\pi) + B_a(2\pi)\varepsilon + O(\varepsilon^2). \tag{15}$$

Comparison of (15) and (11), with the aid of (13), gives the result

$$L_a = B_a(2\pi). \tag{16}$$

The stage has been set for the determination of fixed points of M with the aid of second map C, called a contraction map, defined in terms of M. The map C is defined by requiring that its action on the arbitrary point $\vec{a}$ be given by the rule

$$C\vec{a} = \vec{a} - (I - L_a)^{-1}(\vec{a} - M\vec{a}). \tag{17}$$

Now let $\vec{e}$ be an arbitrary point in the vicinity of a fixed point $\vec{f}$. The contraction map C has the remarkable property that

$$\vec{f} = \lim_{n\to\infty} C^n\vec{e}. \tag{18}$$

That is, a guess as to the whereabouts of a closed orbit is sufficient starting information to contract in on it exactly. In practice, the starting guess can be taken to be the initial conditions for the on-momentum design orbit, and only a few applications of C are required to find a closed orbit to an accuracy of 10 significant figures. The construction of C is based on Newton's method, and the convergence is extremely fast because the error is squared with each application of C.

Determination Of Tunes

Suppose a closed orbit has been found. Let B_f be the matrix solution to the variational equations around this orbit. Then the betatron functions can be computed from $B_f(\theta)$, and the tunes are related to the eigenvalues of $L_f = B_f(2\pi)$. In particular, let $P(\lambda)$ be the characteristic polynomial associated with L_f by the formula

$$P(\lambda) = \det(L_f - \lambda I). \tag{19}$$

Then the two betatron oscillation phase advances $\psi_\pm$ for one lattice circuit are given by the relations

$$\psi_\pm = \cos^{-1}[-b \pm (b^2 - c)^{1/2}] \tag{20}$$

where the quantities b and c are defined by the equations

$$b = [P(1) - P(-1)]/16$$
$$c = -1 + [P(1) + P(-1)]/8. \quad (21)$$

Correspondingly, the associated tunes $T_\pm$ are given by the formulas

$$T_+ = \pm (\psi_+/2\pi) + \text{some integer}$$
$$T_- = \pm (\psi_-/2\pi) + \text{some integer} \quad (22)$$

The integer ambiguity in the tunes can be removed if the lattice is periodic, and the phase advances per lattice period are less than 2π . It is of interest to note that the relations just given make no assumption of mid-plane symmetry, and hold in the general case.

Applications

The general methods outlined in the previous sections have been applied to two specific lattices which are under consideration for a Proton Storage Ring to be built at the Los Alamos National Laboratory. The lattice will have ten identical periods. The elements of each period are listed below:

Element	Path Length or Angle
drift	2.28646 m
hor defocus quad	.5 m
drift	.45 m
edge	0° (or 18°)
bend*	2.54948 m
edge	0° (or 18°)
drift	.45 m
hor defocus quad	.5 m
drift	2.28646 m
Total	9.02240 m

*1.2 Tesla field and design momentum given by BRHO = 4.8691481 Tesla meters.

One proposed lattice has normal entry bend magnets, and the other has parallel faced bend magnets.

Let p^0 denote the design momentum, and let p be a momentum of interest given by the relation

$$p = p^0(1 + \delta). \quad (23)$$

Also, suppose a tune T is expanded in the form

$$T(\delta) = T(0) + \delta T'(0) + \ldots \quad (24)$$

Here T'(0) measures the linear chromaticity. Then the following results are found for the lattice just described:

Normal Entry Case

Horizontal Defocus Quad Strength = -2.68 T/m
Horizontal Focus Quad Strength = 1.95 T/m
$T_{horiz}(0)$ = 2.2540596
$T_{vert}(0)$ = 2.2499258
$T_{horiz}'(0)$ = -1.0762
$T_{vert}'(0)$ = -1.2847

Parallel Faced Case

Horizontal Defocus Quad Strength = -1.92 T/m
Horizontal Focus Quad Strength = 2.72 T/m
$T_{horiz}(0)$ = 2.2541028
$T_{vert}(0)$ = 2.2554377
$T_{horiz}'(0)$ = - .92815
$T_{vert}'(0)$ = -2.11135

It is evident that both natural chromaticities are negative. Provisions have been made for sextupole elements in the lattice to make the chromaticities more negative or to bring them to zero. Detailed numbers, with an excessive number of significant figures, will be presented elsewhere in order to provide benchmark results for comparison with other methods of computation.[1]

As a result of making a series of calculations, it is found that the natural chromaticity of a small ring can vary widely over the tune diagram; and, contrary to common lore, can even be positive. It is also found that nonlinear dipole contributions can be very important for small rings. Consequently, methods of chromaticity calculation which treat dipoles in the linear transfer matrix approximation are not expected to be correct for small rings.

Finally, preliminary comparisons indicate good agreement with the program DIMAT. The program DIMAT makes use of TRANSPORT[2] which retains all non-linear terms in the equations of motion through order two. Thus it appears that second order terms (which are omitted by linear transfer matrix approximations) can be very important for small rings; and third order and still higher order terms (which are neglected by TRANSPORT) need not be important even for small rings.

References

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2. K. L. Brown, SLAC-75, revision 3, (1975); K. L. Brown, F. Rothacker, D. C. Carey, and Ch. Iselin, SLAC-91, revision 2, (1977).