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UNCOUPLED AND COUPLED ORBIT MOTION IN A STORAGE RING.

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Summary.

Using Hamiltonformalism with action and angle variables, uncoupled and coupled orbit motion of particles in a storage ring are presented. Analytical expressions for the Twiss parameters, using Fourier components of the guide field, are deduced from the linear Hamiltonfunction. With these lattice functions, expressions which represent influences of non-linearities are given. The description of coupled motion is, after some canonical transformations, reduced to a one-dimensional problem. The motion is represented in a phase-plane, which has already been done for AVF cyclotrons.[1,2]

The theory will be applied on the Dutch proposed synchrotron radiation source PAMPUS.[3,4]

The results of the linear theory as given here are compared with matrixcalculations. Results for non-linear resonances are compared with results obtained by using the theory of Guignard.[5]

Introduction.

The orbit motion in a storage ring is deduced from a general Hamiltonian.[4,5,6] Interested in the Twiss parameters we first consider the linear (radial) betatronmotion, described by the Hamiltonian [6]

$$H = \tfrac12\bar p_x^2 + \tfrac12(\varepsilon^2(\theta) - n(\theta))\bar x^2 \qquad (1)$$

with the reduced variables $\bar x = x/R$ and $\bar p_x = p_x/p$; $\varepsilon = R/\rho$ and $n = -(R^2/B_o\rho)(\partial B_z/\partial x)_o$ where R is the mean radius and ρ the radius of curvature. The azimuth θ is the independent variable.

It is convenient to use action and angle variables J,ψ defined by

$$\bar x = \sqrt{2J/Q}\cdot\cos(\psi - Q\theta) \text{ and } \bar p_x = \sqrt{2JQ}\cdot\sin(\psi - Q\theta) \qquad (2)$$

where Q is the radial tune.

The new Hamiltonian now becomes

$$K(J,\psi) = H(\bar x,\bar p_x) - QJ = e_2 J + g_2 J \qquad (3)$$

where e_2 is the constant part and g_2 the oscillating part. This oscillating part can be removed by a transformation generated by the function

$$G_1(\bar J,\psi,\theta) = -\bar J\psi - \bar J U_2(\psi,\theta) \qquad (4)$$

In general we can set [7,10]

$$U_2(\psi,\theta) = \sum_{k=1} a_{2k}(\theta)\cos 2k(\psi - Q\theta) + b_{2k}(\theta)\sin 2k(\psi - Q\theta) \qquad (5)$$

The coefficients a_{2k} and b_{2k} have to be determined by the requirement of vanishing oscillating parts and contain power series of the Fouriercomponents of the guide field ($\varepsilon^2 - n$);[7] a_{2k} and b_{2k} have the same periodisity as the guide field.

Furthermore we note the relation between the action variable $\bar J$ and the "emittance of a particle" ε_x :

$$\bar J = \tfrac12\varepsilon_x/R \qquad (6)$$

Considering the common phase-space ellipse (x,dx/ds) one can deduce the relations [7]

$$\sqrt{\varepsilon_x/\beta_x} = \sqrt{2Q\bar J}\left(\sqrt{1+\partial U_2/\partial\psi}\right)_{\psi-Q\theta=\pi/2} \qquad (7)$$

$$\sqrt{\varepsilon_x/\gamma_x} = R\sqrt{2\bar J/Q}\left(\sqrt{1+\partial U_2/\partial\psi}\right)_{\psi-Q\theta=0} \qquad (8)$$

With eq.(5) the lattice functions, expressed in Fourier components of the guide field, are given by

$$\beta_x = (R/Q)\cdot\left(1+\sum_{k=1}(-1)^k 2kb_{2k}\right)^{-1} \qquad (9)$$

$$\gamma_x = (Q/R)\cdot\left(1+\sum_{k=1} 2kb_{2k}\right)^{-1} \qquad (10)$$

and furthermore $\alpha_x^2 = \beta_x\gamma_x - 1$.

The behaviour of the "analytical calculated betatronfunction for the PAMPUS lattice [4] is shown in figure 1. Comparing this with the result of a matrixcode [4] one sees that there is a very good correspondence.

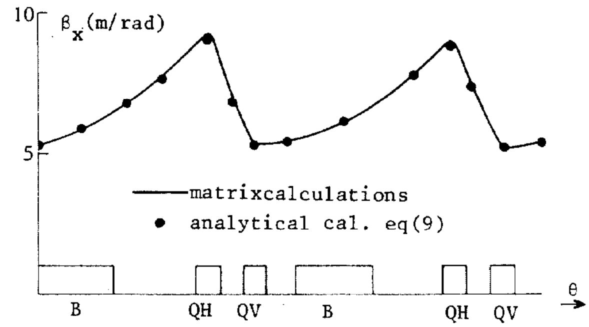

Figure 1. Behaviour of radial betatronfunction for PAMPUS with Q=2.10

Non-linear uncoupled orbit motion.

Non-linear fields often give rise to resonances, which may limit the stability region. Considering the one-dimensional, radial, motion, the Hamiltonian becomes

$$H = \tfrac12\bar p_x^2 + \tfrac12(\varepsilon^2 - n)\bar x^2 + h(\bar x,\bar p_x,\theta) \qquad (11)$$

This Hamiltonian can be rewritten as:

$$\bar{\bar H} = \tfrac12\bar{\bar p}_x^2 + \tfrac12 Q^2\bar{\bar x}^2 + \sum_{j,k} a_{j,k}(\phi)\bar{\bar x}^j\bar{\bar p}_x^k \qquad (12)$$

where j and k are positive integers($j+k\geq 3$); n=j+k is the degree of the Hamiltonian. Furthermore the phase ϕ ($d\phi = R/Q\beta d\theta$) is the independent variable and

$\bar{\bar x} = \sqrt{\beta/R}\cdot\bar x$, $\bar{\bar p}_x = \sqrt{R/\beta}(\bar p_x - \alpha\bar x)$, where we omitted the index at the lattice functions. We introduce action and angle variables,

$$\bar{\bar x} = \sqrt{2I/Q}\cos\psi \ ,\ \bar{\bar p}_x = \sqrt{2QI}\sin\psi \text{ with } I = \tfrac12 Q\varepsilon_x/R \qquad (13)$$

The non resonant oscillating parts of n^{th} degree in the Hamiltonian can be removed, using a transformation, generated by a function of the form [6]

$$G_2(\bar I,\psi,\phi) = -\bar I\psi - \bar I^{n/2}U_n(\psi,\phi) \qquad (14)$$

The new Hamiltonian, containing resonant terms, now becomes:

$$K(\bar I,\bar\psi) = Q\bar I + \sum_{m_r}\sum_{p_r} F_{n,p_r}^{(m_r)} e^{i(m_r\bar\psi + p_r N\phi)}\bar I^{n/2} \qquad (15)$$

The function $F_n^{(m)}(\phi)$ is the coefficient of a term of n-th degree and argument $m\psi$; m can take the values $m = \pm n, \pm(n-2),\ldots$. Expanding $F_n^{(m)}$ in a Fourierseries delivers the quantity $F_{n,p}^{(m)}$.

The p_r harmonic of $F_n^{(m)}(\phi)$ drives the resonances $m_r Q = p_r N$ (note that in the linear case $d\bar\psi/d\phi = -Q$). N is the periodicity of the non-linearity.

The ϕ-dependence is now removed by a transformation

$$\bar{\bar I} = \bar I \text{ and } \bar{\bar\psi} = \bar\psi + (p_r/m_r)N\phi \qquad (16)$$

which is generated by the function G_3 :

$$G_3(\bar{\bar I},\bar\psi,\phi) = -\bar{\bar I}\bar\psi - (p_r/m_r)N\phi\bar{\bar I} \qquad (17)$$

The Hamiltonian can now be written as [6]

$$\bar K(\bar{\bar I},\bar{\bar\psi}) = (Q - \frac{p_r}{m_r}N)\bar{\bar I} + 2|F_{n,p_r}^{(m_r)}|\cos(m_r\bar{\bar\psi} + \chi_F)\bar{\bar I}^{n/2} \qquad (18)$$

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where m_r and p_r are now both positive, and

$$F^{(m_r)}_{n,p_r} = |F^{(m_r)}_{n,p_r}| e^{i\chi_F} \quad (19)$$

The stability region can be predicted from the invariant $\bar{\bar{K}}(\bar{\bar{I}},\bar{\bar{\psi}})$. The criterion for the fixed points ($d\bar{\bar{I}}/d\phi = 0$, $d\bar{\bar{\psi}}/d\phi = 0$) leads to the condition

$$\bar{\bar{I}}_{f.p.}^{n/2-1} = \pm(Q - \frac{p_r}{m_r} N)/(n|F^{(m_r)}_{n,p_r}|) \quad (20)$$

where $\bar{\bar{I}}_{f.p.}$ is the action variable $\bar{\bar{I}}$ belonging to the unstable fixed points.

A criterion for stability is that the beam lies entirely inside the stable region in the $(\bar{\bar{I}},\bar{\bar{\psi}})$ phase plane. Representing the beam by a circle (correct when no non-linearities are present) one gets the requirement [6]

$$\bar{\bar{I}}_{f.p.} \geq c_n(Q/R)\varepsilon_x \quad (21)$$

where c_n is a constant depending on the form of the stable region and so depending on n [6] : c_3=2.0 and c_4= 1.21 . Eqs.(20) and (21) lead, for a given excitation term, to a minimum distance to the resonance line.

We will illustrate this for the third degree resonance $3Q=p_rN$ excited by sextupole fields. The minimum distance is given by [6]

$$|Q - \frac{p_r}{3}\cdot N| = 3|F^{(3)}_{3,p_r}|(2Q\varepsilon_x/R)^{\frac{1}{2}} \quad (22)$$

with $F_3^{(3)}=(\sqrt{2}/24)(\beta^{5/2}Q^{1/2}R^{1/2})(\partial^2B_z/\partial x^2)/(B_o\rho)$

The sextupoles needed for chromaticity correction in *PAMPUS* [4] (N=8) can for example excite the resonance 3Q=16 . For a particle with a 10 σ-amplitude, the minimum distance is [6] $|Q-16/3| = 0.036$.

Furthermore we note the $\bar{\bar{\psi}}$ and ϕ independent Hamiltonian of eq.(15) for m_r=0 , p_r=0 . This condition can only occur for even degree n. In this case there is a change of the tune, depending on the amplitude. Considering the fourth degree Hamiltonfunction the tune-shift is given by

$$\Delta Q = 2\ F^{(0)}_{4,0}\ \bar{\bar{I}} \quad (23)$$

with $F_4^{(0)}= (3/16)R\beta\gamma^2 + (1/16)\beta^3R(\partial^3B_z/\partial x^3)/(B_o\rho)$

The first term gives rise to an inherent tune shift and the second term is the contribution of octupole fields. The needed octupole fields to provide a certain tune shift(spread) can be determined. As an example [6] : for PAMPUS at Q = 3.25 the octupole field should be $\partial^3B_z/\partial x^3) = 100\ T/m^3$ in order to get a tune shift of 10^{-3} for a particle with a 10 σ-amplitude.

Non-linear coupled motion

Two transverse motions may be coupled in the presence of non-linearities. We will consider here the influences of sextupoles and octupoles and the Hamiltonian of interest is

$$H = \tfrac{1}{2}\bar{p}_x^2 + \tfrac{1}{2}(\varepsilon^2-n)\bar{x}^2 + \tfrac{1}{2}\bar{p}_z^2 + \tfrac{1}{2}n\bar{z}^2 + \sum_{j,l} a_{j,l}(\theta)\bar{x}^j\bar{z}^l \quad (24)$$

$\bar{x}$, $\bar{z}$ and $\bar{p}_x$, $\bar{p}_z$ are relative variables according to eq.(1).

The θ-dependence in the linear Hamiltonian is removed by a transformation, generated by the function (see also [9])

$$G_4(\bar{x},\psi_x,\bar{z},\psi_z,\theta) = (R^2/2\beta_x)\bar{x}^2\{\tan(\psi_x+\mu_x(\theta))-\alpha_x\} + \quad (25)$$
$$+ (R^2/2\beta_z)\bar{z}^2\{\tan(\psi_z+\mu_z(\theta))-\alpha_z\}$$

where $\beta_{x,z}$ is the betatronfunction , $\alpha_{x,z}=-\frac{1}{2R}(\frac{d\beta_{x,z}}{d\theta})$ and $\mu_{x,z}= Q_{x,z}\theta - \int_o^\theta(R/\beta_{x,z})d\theta'$

The Hamiltonian can now be written as $K(J_x,\psi_x,J_z,\psi_z,\theta)$. Keeping in this Hamiltonian only the non-linear resonant terms and applying the moving-coordinate transformation of the type as given in eq.(16), which is now generated by the function G_5 :

$$G_5(J_1,\psi_x,J_2,\psi_z,\theta) = -J_1\psi_x - J_2\psi_z - (m_1/m_2)J_2\psi_x - Q_xJ_1\theta + -(p_rN/m_2)J_2\theta \quad (26)$$

with $J_1= J_x-(m_1/m_2)J_z$, $J_2= J_z$, $\psi_2=\psi_z+\frac{m_1}{m_2}\psi_x+\frac{p_rN}{m_2}\theta$

The new Hamiltonian now becomes:[8]

$$\bar{K} = \delta QJ_2 + 2|F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}|(J_1+\frac{m_1}{m_2}J_2)^{\frac{|m_1|}{2}}J_2^{\frac{|m_2|}{2}}\cdot\cos(m_2\psi_2+\chi_F) \quad (27)$$

with $m_2\delta Q = m_1Q_x + m_2Q_z - p_rN$

$F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}$ is the coefficient belonging to a term of degree $|m_1|+|m_2|$ (degree $|m_1|$ in $\bar{x}$ and $|m_2|$ in $\bar{z}$) and p_r the fouriercomponent that drives the resonance. As ψ_1 is not present in this Hamiltonian J_1= constant and the problem of describing the couplingsresonance is reduced to a one-dimensional problem,for which it is now convenient to present it in a phase-plane. The fixed points are given by $dJ_2/d\theta$ =0 and $d\psi_2/d\theta$=0 .

In the following sections we will study several resonances with the add of phase-plane figures. From these figures we can define a minimum δQ value in order to avoid unstable motion. These values will be compared with results obtained by using the theory of Guignard.[5]

$2Q_x + Q_z = p_rN$: skew sextupoles.

This resonance can be excited by skew sextupole fields. The excitation term $F^{(2,1)}_{2,1}$ is [8]

$$F^{(2,1)}_{2,1}=(\sqrt{2}/8)\beta_x\beta_z^{1/2}(R^{3/2}/B_o\rho)(\partial^2B_z/\partial x^2)e^{i(2\psi_x+\psi_z)} \quad (28)$$

The study of phase plane trajectories is somewhat simplified by the transformation

$$\tilde{x} = \sqrt{2J_2}\cos\psi_2 \text{ and } \tilde{y} = \sqrt{2J_2}\sin\psi_2 \quad (29)$$

and the new Hamiltonian is called $\tilde{\bar{K}}$.

The fixed points in $(\tilde{x},\tilde{y})$ phase plane are now given by

$$d\tilde{x}/d\theta= \delta Q\tilde{y} + 2\sqrt{2}|F|\tilde{x}\tilde{y} = 0 \quad (30)$$
$$d\tilde{y}/d\theta= -\sqrt{2}|F|J_1-\delta Q\tilde{x} -3\sqrt{2}|F|\tilde{x}^2 -\sqrt{2}|F|\tilde{y}^2 = 0$$

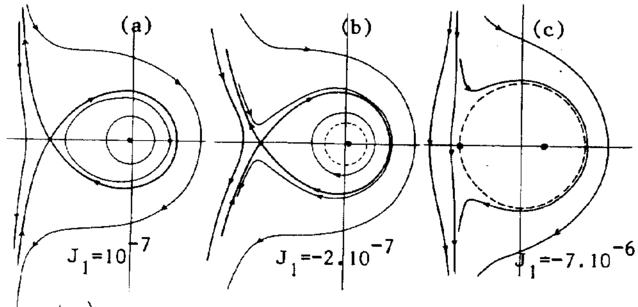
where we omitted the subscripts at $F^{(2,1)}_{2,1,p_r}$ and $\delta Q = 2Q_x + Q_z - p_rN$.

It follows from eq.(30) that the flowlines and the position of the fixed points are related to the value of $J_1= J_x - 2J_z$ and $\delta Q/|F|$. This is illustrated in fig.2 where phase plane trajectories are shown for a fixed $\delta Q/|F|$ value and different J_1 values.[8]

Analogously to the one-dimensional case we are now interested in the minimum δQ value in order to avoid unstable motion. The criterion for this is, that the beam representation (a circle in the linear case) must lie entirely inside the stable region in the $(\tilde{x},\tilde{y})$ phase plane. Since the separatrix is known, the maximum allowable 'beam-emittance' $J_{2,beam}$ can be expressed in coordinates of the unstable fixed point(s) (as done in the one-dimensional case by eq.(21)) which is now a function of J_1 and $\delta Q/|F|$ (see eq.(30)).

Given the horizontal and vertical emittances a minimum value of $\delta Q/|F|$ can be given. The resulting $\delta Q/|F|$ curves in a $J_x= \tfrac{1}{2}\varepsilon_x/R$, $J_z= \tfrac{1}{2}\varepsilon_z/R$ diagram are given in figure 3.

So given a certain machine (ε_x, ε_z, R, F) the minimum δQ value can easily be determined from this figure. An example is shown in fig.3 : a machine with[8] $J_x= 22.10^{-7}$ and $J_z= 7.8\ 10^{-7}$ should satisfy the condition $\delta Q/|F| \geq 10.6\ 10^{-3}$



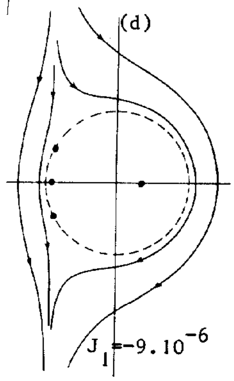


Fig.2 Trajectories in $(\tilde{x},\tilde{y})$ phase space. $\delta Q=0.075$ and $|F|=10$. The region inside the dashed circle (b,c,d) indicate the unphysical region ($J_x<0$).

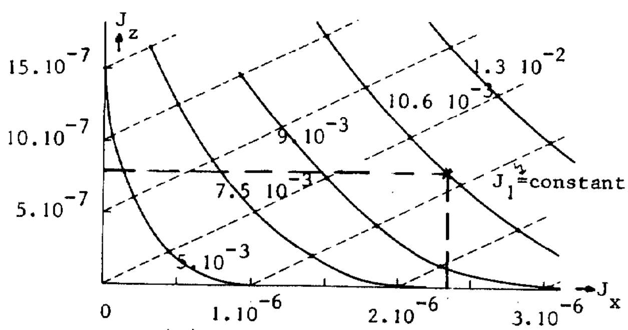


Figure 3. $\delta Q/|F|$ curves in the J_x,J_z diagram for the resonance $2Q_x+Q_z=p_rN$

$Q_x + 2Q_z = p_rN$: normal sextupoles.

The study of this resonance goes in a same way as the previous resonance. The excitation term is now

$$F^{(1,2)}_{1,2}=-(\sqrt{2}/8)\beta_x^{\frac{1}{2}}\beta_z(R^{3/2}/B_o\rho)(\partial^2B_z/\partial x^2)e^{i(\psi_x+2\psi_z)} \quad (31)$$

and furthermore $\delta Q = \frac{1}{2}(Q_x+ 2Q_z- p_rN)$, $J_1= J_x- \frac{1}{2}J_z$

Trajectories in $(\tilde{x},\tilde{y})$ phase plane are given in figure 4.

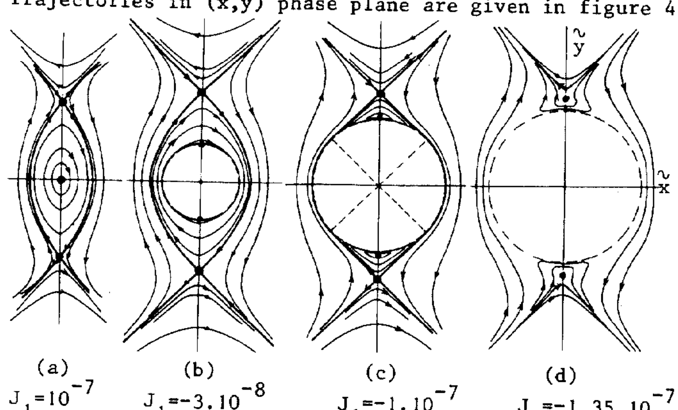


Fig.4 $(\tilde{x},\tilde{y})$ phase plane for PAMPUS for the resonance $Q_x+2Q_z=p_rN$; $\delta Q=0.05$, $|F|=40.1$ [8]

In the same way as done in the previous section we can calculate the minimum $\delta Q/|F|$ value for a given J_x,J_z. The result is given in fig.5 : PAMPUS should satisfy $\delta Q/|F|>1.1\ 10^{-3}$.[8]

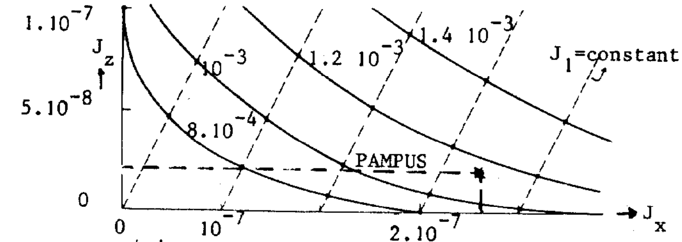


Fig.5 $\delta Q/|F|$ curves as function of J_x,J_z;$\delta Q=\frac{1}{2}(Q_x+2Q_z-p_rN)$

Results of this phase-plane treatment are compared with results, obtained by using the theory of Guignard.[5] The minimum distance given by Guignard is strongly related to the equation which holds for the fixed points and substituting $J_{2\ f.p.}=\frac{1}{2}\varepsilon_z/R$ (see also [8])

resonance	$\|F\|$	δQ(our method)	δQ (Guignard)
$Q_x+2Q_z=8$	0.14	0.00050	0.00039
$Q_x+2Q_z=16$	40.1	0.044	0.034

Table 1. $\delta Q=\frac{1}{2}(Q_x+2Q_z-p_rN)$ values for PAMPUS. [8]

$2Q_x - 2Q_z = 0$: octupoles.

Trajectories in $(\tilde{x},\tilde{y})$ phase-plane are given in fig.6 for fixed values of J_1 and $F^{(2,-2)}_{2,2,0}$ and different values of $\delta Q=\frac{1}{2}(2Q_x-2Q_z)$.

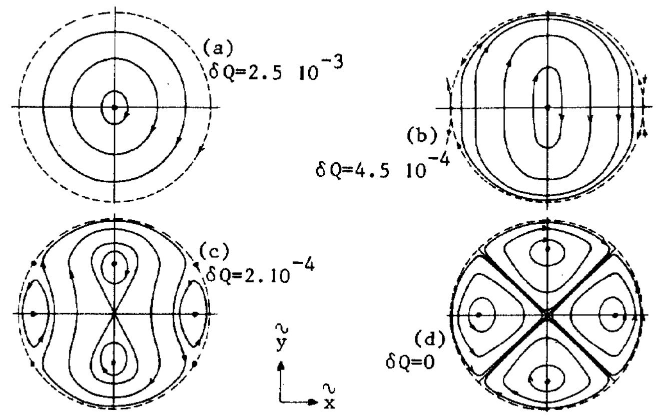


Fig.6 $(\tilde{x},\tilde{y})$ phase-plane with $J_1=9.10^{-7}$, $|F^{(2,-2)}_{2,2,0}|=250$

This resonance leads to a periodic exchange of energy between the two transverse planes. This exchange can be determined by using extreme values ± 1 for $\cos 2\psi_2$ in $\bar{K}$ (eq.27) [8] : $\alpha=J_{2,min}/J_{2,max}$ and

$$\alpha^2 \mp \left(\frac{|\delta Q| +2|F|J_1}{2|F|J_{2,max}}\right)\alpha \pm \left(\frac{|\delta Q| -2|F|J_1}{2|F|J_{2,max}} + 1\right) = 0 \quad (32)$$

The upper sign holds for $\delta Q>0$, the lower sign for $\delta Q<0$.

Final remarks.

The analytical expressions give a good behaviour of the lattice functions. Furthermore the description of non-linear resonances in a one-dimensional phase-space gives a very good insight in the influence of the exciting non-linear fields.

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