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# HIGHER-ORDER TERMS IN A PERTURBATION EXPANSION FOR THE IMPEDANCE OF A CORRUGATED WAVE GUIDE*

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## Abstract

We consider a circular waveguide whose radius $a(z) = a(1+\varepsilon s(z))$ varies periodically with the axial coordinate z. Chatard-Moulin and Papiernik[1] have introduced a perturbation expansion in powers of $\varepsilon$ for the longitudinal impedance of such a waveguide. We have reformulated the derivation of this expansion in a manner which elucidates the structure of the higher-order terms, and allows the determination of the dependence on $\varepsilon$ of the resonant frequencies. For a square-wave (SW) wall distortion, there are divergent coefficients in the perturbation expansion. Hence, a special treatment is required in this case, and we present a calculation of the resonant behavior for small $\varepsilon$, using an approach which does not assume an expansion in powers of $\varepsilon$. The resulting expression for the resonant impedance involves functions singular at $\varepsilon=0$; however, to leading order in $\varepsilon$, the loss-factors and resonant frequencies are in agreement with the perturbation theory of ref. 1.

## Derivation of the Perturbation Expansion

Chatard-Moulin and Papiernik[1] have introduced a perturbation technique for the calculation of the electromagnetic fields generated by an electron beam moving along the axis of a circular waveguide, whose radius a(z) varies periodically with the axial coordinate z. The perturbation parameter $\varepsilon$ is introduced via

$$a(z) = a(1+\varepsilon s(z)), \tag{1}$$

$$s(z) = \sum_{p=-\infty}^{\infty} C_p \exp(2\pi i p z/L), \tag{2}$$

where a denotes the unperturbed radius, and the shape function $s(z) = s(z+L)$ has period L. Applications of the method developed in ref. 1 can be found in the work of Krinsky[2] and Cooper and Morton[3]. Here, we reformulate the derivation of the perturbation expansion to elucidate the structure of the higher-order terms[4].

We begin by assuming an axial current density,

$$J_z=\left(I(\omega)/\pi\rho^2\right)\theta(\rho-r)\exp(-i\omega\tau),\quad \tau=t-z/v, \tag{3}$$

where cylindrical coordinates $(r,\phi,z)$ are employed. The electron beam radius is denoted $\rho$, the phase velocity v, the laboratory time t and the step function $\theta(x)$ vanishes for $x<0$ and is unity for $x>0$. The current density generates an azimuthal magnetic field, $H_\phi(r,z)\exp(-i\omega\tau)$, where $H_\phi(r,z+L) = H_\phi(r,z)$ is determined by the inhomogeneous wave equation[1] ($\gamma^{-2}=1-v^2/c^2$):

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r}\right) - \frac{\omega^2}{v^2\gamma^2}H_\phi+\frac{\partial^2 H_\phi}{\partial z^2}+\frac{2i\omega}{v}\frac{\partial H_\phi}{\partial z}=\frac{\partial J_z}{\partial r}, \tag{4}$$

together with the boundary condition[1] at $r = a(z)$:

$$\frac{\partial(rH_\phi)}{\partial r} = a'(z)\left[\frac{\partial(rH_\phi)}{\partial z}+\frac{i\omega}{c}(rH_\phi)\right]. \tag{5}$$

Here, $a'(z) = da(z)/dz$, and the partial derivative $\partial/\partial z$ is taken with $\tau = t - z/v$ held fixed. The electric fields $E_z$ and $E_r$ can be calculated from $H_\phi$ using the curl equations.

For simplicity, we consider the limit $\gamma\to\infty$, so that $v \approx c$ and $\omega/c\gamma \ll 1$. Then the general solution to Eq. (4) is

$$H_\phi(r,z)/I(\omega)=g(r) - \frac{\omega}{cL}\sum_{p=-\infty}^{\infty} B_p \frac{I_1(\Lambda_p r)}{\Lambda_p I_o(\Lambda_p a)}e^{2\pi i p z/L}, \tag{6}$$

where $g(r) = r/2\pi\rho^2 (r\le\rho)$, and $g(r) = 1/2\pi r\ (r>\rho)$,

$$\Lambda_p^2 = (2\pi p/L)^2 + (2\pi p/L)(2\omega/c), \tag{7}$$

and $I_m$ is the modified Bessel function. In order to derive an infinite set of linear equations for the unknown coefficients $B_p$, we substitute (6) into the boundary condition (5). It is useful to define

$$\Gamma_{np}=(2\pi n/L)(\omega/c)+(2\pi p/L)(\omega/c)+(2\pi n/L)(2\pi p/L), \tag{8}$$

$$D_k(\Lambda_p)=\frac{1}{L}\int_0^L dz\ \exp(-2\pi i k z/L)I_o(\Lambda_p a(z))/I_o(\Lambda_p a) \tag{9}$$

$$= \delta_{ko}+\Lambda_p^2 \sum_{m=1}^{\infty} Q_p^{(m)} C_k^{(m)} \varepsilon^m, \tag{9a}$$

$$E_k= \frac{1}{L}\int_0^L dz\ \exp(-2\pi i k z/L)\ a'(z)/a(z) \tag{10}$$

$$= -(2\pi i k/L)\sum_{m=1}^{\infty} (-)^m C_k^{(m)} \varepsilon^m/m, \tag{10a}$$

where $C_k^{(m)}$ is the k-th Fourier coefficient of $[s(z)]^m$,

$$Q_p^{(m)}= \left(a^m\Lambda_p^{m-2}/m!\right) I_o^{(m)}(\Lambda_p a)/I_o(\Lambda_p a), \tag{11}$$

and $I_o^{(m)}$ is the m-th derivative of $I_o$. It now follows that the coefficients $B_p$ are determined by:

$$\sum_{p=-\infty}^{\infty} B_p D_{n-p}(\Lambda_p)\Gamma_{np}/\Lambda_p^2 = -iLE_n/2\pi\ (n=-\infty,\ldots,\infty). \tag{12}$$

For vanishing electron beam radius, $\rho\to o$, the longitudinal impedance, $Z(\omega)$, is related to the solution of Eq. (12) by

$$Z(\omega) = iZ_o B_o, \tag{13}$$

where $Z_o = 1/c\varepsilon_o$.

Using Eqs. (9a), (10a) and (12), a perturbation expansion is derived[4]:

$$B_p = \sum_{m=1}^{\infty} B_p^{(m)}\varepsilon^m, \tag{14}$$

$$B_p^{(m)}= U_p^{(m)}+\sum_{m'=1}^{m-1}\sum_{p'=-\infty}^{\infty} G_{pp'}^{(m')} B_{p'}^{(m-m')}, \tag{15}$$

with $U_p^{(m)}= (-)^{m+1}(p/m)\ C_p^{(m)}$ and $G_{pp'}^{(m)}=-\Gamma_{pp'}C_{p-p'}^{(m)}Q_{p'}^{(m)}$.

Now defining $b(m) = U_o^{(m)} = 0$, and for $k\ge 2$,

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$$b(m_1,\ldots,m_k) = \sum_{p_1,\ldots,p_{k-1}} G^{(m_1)}_{op_1} G^{(m_2)}_{p_1p_2} \cdots G^{(m_{k-1})}_{p_{k-2},p_{k-1}} U^{(m_k)}_{p_{k-1}}, \tag{16}$$

it is easily seen that

$$B_o^{(m)} = \sum_{(m)} b(m_1,\ldots,m_k), \tag{17}$$

where the sum $\Sigma_{(m)}$ is over the $2^{m-1}$ sets of integers $[m_j]$, with $m_j \geq 1$ and $\Sigma m_j = m$. In particular, $B_o^{(2)} = b(1,1)$, and $B_o^{(3)} = b(2,1) + b(1,2) + b(1,1,1)$.

The $O(\varepsilon^2)$ contribution to the impedance is

$$Z(\omega)/iZ_o = -(2\pi\omega/cL)\sum_{p=-\infty}^{\infty}|\varepsilon p C_p|^2 Q_p, \tag{18}$$

where we have introduced $Q_p \equiv Q_p^{(1)}$. The resonant frequencies[1-3], therefore, correspond to the poles of $Q_p$, i.e. $I_o(\Lambda_p a) = 0$. If at a given frequency $\bar{\omega}$, one has $Q_{\bar{p}}$ nearly infinite and dominating all other $Q_p(p\neq\bar{p})$, then in the neighborhood of $\bar{\omega}$ we might expect,

$$Z(\omega)/iZ_o \simeq \left(\sum_{m=2}^{\infty}\nu_m(\bar{p})\varepsilon^m\right) / \left(Q_{\bar{p}}^{-1} - \sum_{m=1}^{\infty}\delta_m(\bar{p})\varepsilon^m\right). \tag{19}$$

Comparing coefficients of $(Q_{\bar{p}})^{\ell}\varepsilon^n$ $(1\leq\ell<n-1)$ in Eqs. (17) and (19), one determines $\nu_m(\bar{p})$ and $\delta_m(\bar{p})$. Clearly, $\nu_m(\bar{p})$ is just the coefficient of $Q_{\bar{p}}\,\varepsilon^m$. To write an expression for $\delta_m(p)$, we define

$Q_p\Delta_p(m) = G^{(m)}_{pp}$, and for $k \geq 2$,

$$Q_p\Delta_p(m_1,\ldots,m_k) = \sum G^{(m_1)}_{pp_1} G^{(m_2)}_{p_1p_2} \cdots G^{(m_{k-1})}_{p_{k-2},p_{k-1}} G^{(m_k)}_{p_{k-1}p}, \tag{20}$$

where the sum is over $p_1 \neq p, \ldots, p_{k-1} \neq p$. Then, we find

$$\delta_m(p) = \Sigma_{(m)}\Delta_p(m_1,\ldots,m_k). \tag{21}$$

As we have done before in Eq. (17), we use $\Sigma_{(m)}$ to denote the summation over the $2^{m-1}$ sets of integers $[m_j]$, with $m_j \geq 1$ and $\Sigma m_j = m$. To be explicit, we write $\delta_1(p) = \Delta_p(1)$, $\delta_2(p) = \Delta_p(2) + \Delta_p(1,1)$, and $\delta_3(p) = \Delta_p(3) + \Delta_p(2,1) + \Delta_p(1,2) + \Delta_p(1,1,1)$. Knowledge of $\delta_m(p)$ determines the variation of the resonant frequencies with $\varepsilon$, as long as Eq. (19) is a valid approximation, that is, until the resonant frequency corresponding to some $p \neq \bar{p}$ becomes degenerate with that corresponding to $\bar{p}$.

## Square Wave Wall Distortion

Let us now consider the square wave (SW) perturbation of the wall studied by Keil and Zotter.[5] We take the shape function $s(z)$ equal to unity for $0<z<g$, and to zero for $g<z<L$. Its Fourier coefficient,

$$C_p^{SW} = (2\pi ip)^{-1}[\exp(2\pi ipg/L)-1] \tag{22}$$

falls off slowly at large p, resulting in the divergence of the coefficients in the perturbation expansion (14). This divergence indicates that the impedance considered as a function $\varepsilon$ is singular at the origin for the SW perturbation.

In order to make sense of the perturbation expansion, it is necessary to introduce a large p cutoff $p_{max}$ and to perform a partial summation including leading terms in $p_{max}$ for all orders of $\varepsilon$. After completing the summation, $p_{max}$ is allowed to go to infinity and a finite result is obtained. This procedure has been applied in ref. 4 to treat the low frequency limit of $Z(\omega)/\omega$. For a narrow cavity with $g\ll L$, $g\ll a\varepsilon$, $\varepsilon\ll 1$, the well-known linear behavior $Z(\omega)/\omega \approx -iZ_o g\varepsilon/2\pi c$ was obtained. For a cavity with g/L of order unity, examination of the higher-order terms indicated that $Z(\omega)/\omega = (-iZ_o a\varepsilon^2/\pi^2 c)\,\ln(L/a\varepsilon) + O(\varepsilon^2)$.

Here we shall present a more direct treatment[6] of the SW distortion, focussing our attention on the resonant behavior at high frequencies. Our objective shall be to make contact with the perturbation theory results of Eqs. (19 -21). In the "tube region" $(0 \leq z \leq L,\ r \leq a)$ we express the EM-fields in terms of coefficients $B_p$ using Eq. (6). In the "cavity region" $(0 \leq z \leq g,\ a \leq r \leq b \equiv a(1+\varepsilon))$, the fields are written in terms of coefficients $\beta_n$,

$$H_\phi(r,z,t)/I(\omega) = \sum_{n=o}^{\infty}\beta_n[T_1(\psi_n r)/\psi_n T_o(\psi_n a)]f_n(z)\exp(-i\omega t). \tag{23}$$

The functions $f_n(z) = (2/g)^{1/2}\cos(n\pi z/g)$, $n \geq 1$, and $f_o(z) = (1/g)^{1/2}$ are orthonormal on $0 \leq z \leq g$. We have defined,

$$\psi_n^2 = (n\pi/g)^2 - (\omega/c)^2, \tag{24}$$

and $T_k(\psi_n r) = (-)^k K_k(\psi_n r) - I_k(\psi_n r)K_o(\psi_n b)/I_o(\psi_n b)$, for $k = 0, 1$.

The relation between $B_p$ and $\beta_n$ follows from continuity of $E_z$ along $(0 \leq z \leq g,\ r = a)$:

$$B_p = (-c/\omega)\sum_{n=o}^{\infty}\beta_n R_{np}, \tag{25}$$

where

$$R_{np} = \int_o^g dz\, f_n(z)\exp(-i\omega z/c - 2\pi ipz/L). \tag{26}$$

Continuity of $H_\phi$ results in an equation determining $\beta_n$,

$$S_m^{-1}\beta_m - (aL)^{-1}\sum_{p=-\infty}^{\infty}Q_p R^*_{mp}\sum_{n=o}^{\infty}\beta_n R_{np} = (2\pi a)^{-1}R^*_{mo}, \tag{27}$$

with $S_m = \psi_m T_o(\psi_m a)/T_1(\psi_m a)$. Once $\beta_n$ is known, the impedance is calculated from Eqs. (13) and (25). The relation of Eq. (27) to a variational principle for the impedance will be discussed in the Appendix.

As in the analysis leading to Eq. (19), let us suppose that at a frequency $\bar{\omega}$, a single $Q_{\bar{p}}$ is much larger than all other $Q_p$ $(p\neq\bar{p})$. Then neglecting all $p\neq\bar{p}$ in the summation over p in Eq. (27), we are left with a separable equation, which is easily solved, yielding

$$\frac{Z(\omega)}{iZ_o} = \frac{-c/\omega}{2\pi a}\left[\sum_{n=o}^{\infty}S_n|R_{no}|^2 + \frac{\left|\sum_{n=o}^{\infty}S_n R_{no}R^*_{n\bar{p}}\right|^2}{aLQ_{\bar{p}}^{-1} - \sum_{n=o}^{\infty}S_n|R_{n\bar{p}}|^2}\right]. \tag{28}$$

The summations in Eq. (28) define functions of $\varepsilon$ singular at the origin. To see that the sums are convergent, note that for fixed $\varepsilon$, and $\psi_n a \to \infty$, one has $S_n \approx -\psi_n \tanh(\psi_n a\varepsilon)$, and that for large n, $R_{np}$ decreases quadratically.

From Eq. (28), we can see that the perturbation expansion of Eqs. (14-21) corresponds to an invalid interchange of the order of the summation over powers of $\varepsilon$, and the summation over n. Expanding $S_n$ (for n fixed) in powers of $\varepsilon$, the coefficients of all the higher-order terms are divergent. However, the lowest-order terms in the numerator and denominator are finite and in agreement with perturbation theory. To see this, note that for fixed n, and $\varepsilon \to o$, one has $S_n \approx -\psi_n^2 a\varepsilon$, and use:

$$\sum_{n=o}^{\infty} \psi_n^2 R_{no} R_{np}^* = 2\pi(\omega/c)pC_p^{SW} \quad , \tag{29}$$

$$\sum_{n=o}^{\infty} \psi_n^2 \left|R_{np}\right|^2 = g\Lambda_p^2 \quad , \tag{30}$$

where $C_p^{SW}$ was defined in Eq. (22) and $\Lambda_p^2$ in Eq. (7).

At the present time, we have not accomplished the full asymptotic analysis for $\varepsilon \to o$ of the sums appearing in Eq. (28). We do believe, however, that the proper treatment of the large n portion of the summations will not change the leading terms just presented, which agreed with perturbation theory. An indication of the nature of the singularity at $\varepsilon = o$ is obtained by using the approximation $S_n \approx -\psi_n \tanh(\psi_n a\varepsilon)$ for large n, and converting the summation to an integral. In this manner one finds the singular term $\varepsilon^2 \mathrm{sgn}(\varepsilon)$.

## Sinusoidal Wall Distortion

For a sinusoidal wall distortion, corresponding to a shape function $s(z) = \cos(2\pi z/L)$, the coefficients of all powers of $\varepsilon$ in the perturbation expansion (14) are finite. By iterating the recursion relations (15) on the computer, we have numerically evaluated the coefficients in the expansion of $Z(\omega)/\omega$ in the low frequency limit. Depending on the value of the ratio a/L, we have obtained between 20 and 30 coefficients, before encountering a loss of precision. These numerical results indicate that $\varepsilon = o$ is a regular point, and that the singularities nearest to the origin, determining the radius of convergence, are located at

$$(2\pi a\varepsilon/L)^2 = -1 \ . \tag{31}$$

Pade' approximants[7] have been used to analytically continue the perturbation expansion into the entire interval $o \le \varepsilon \le 1$. In Fig. 1, we plot the ratio of the [10/10] Pade' approximant to the result of second-order perturbation theory.

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## Appendix - Variational Principle

Define

$$\beta(z) = \exp(-i\omega z/c) \sum_{n=o}^{\infty} \beta_n f_n(z) \ , \tag{A1}$$

then Eq. (27) is equivalent to the integral equation

$$\int_o^g dz' \ K(z,z') \ \beta(z') = (2\pi a)^{-1} \ , \ (0 \le z \le g) \tag{A2}$$

with the Hermitean kernel

$$K(z,z') = \exp\left[-i\omega(z-z')/c\right] \sum_{n=o}^{\infty} S_n^{-1} f_n(z) \ f_n(z') \tag{A3}$$

$$-(aL)^{-1} \sum_{p=-\infty}^{\infty} Q_p \exp\left[2\pi i p(z-z')/L\right] \ .$$

From Eqs. (25) and (13), the impedance is given by

$$Z(\omega)/iZ_o = -(c/\omega) \int_o^g dz \ \beta(z) \ . \tag{A4}$$

Now using the integral equation (A2), we can rewrite this as

$$Z(\omega)/iZ_o = -\frac{c/\omega}{2\pi a} X^{-1} \quad , \tag{A5}$$

with

$$X = \frac{\int_o^g \int_o^g dz \ dz' \ \beta^*(z) K(z,z') \ \beta(z')}{\int_o^g \beta^*(z) \ dz \int_o^g \beta(z') \ dz'} \tag{A6}$$

The original integral equation (A2) is equivalent to the variational condition $\delta X = 0$, when $\beta(z)$ and $\beta^*(z)$ are varied independently. Hence, if one has an approximate solution to the integral equation, the error in the impedance is quadratic in the deviation of $\beta(z)$ from the exact solution.

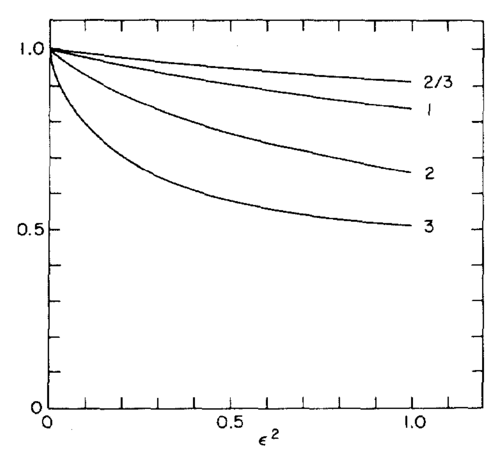


Figure 1. Ratio of the [10/10] Pade' approximant for $Z(\omega)/\omega$ ($\omega \to o$) to the second-order result, for $2\pi a/L$ = 2/3, 1, 2, and 3.