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TURBULENT BUNCH LENGTHENING AND THE MICROWAVE INSTABILITY

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Summary

Including negative mode numbers in the analysis of bunched beam oscillations results in a quadratic equation for the coherent frequency shift. This leads to complex solutions above a threshold, and hence to instability. Bunch lengthening occurs in order to reestablish equilibrium.

Introduction

In most high energy particle accelerators or storage rings one observes an increase of the bunch length with increasing current. For low currents, this bunch lengthening can be explained by the deformation of the potential well due to space charge in an essentially inductive environment. However, above a certain threshold, the rate of bunch lengthening increases and is accompanied by an increase of the bunch width (energy spread) which cannot be caused by a conservative potential. This behaviour has been explained tentatively by the onset of longitudinal oscillations at very high frequencies, which are often called "microwave instability". When the bunches oscillate in many modes and their signals become fluctuating, the phenomenon is called "turbulence" in analogy to the flow of fluids.

In spite of this qualitative understanding [1,2] and an appreciable number of attempts to explain this behaviour quantitatively [3-5] there still exists no fully satisfactory theory which can predict thresholds and bunch lengths for a given wall impedance. The best results for long bunches are obtained by using the coasting beam criterion with local values of beam current and energy spread [6] and using a low-Q resonant impedance adapted to the (measured) low frequency inductance. For shorter bunches, a "scaling model" [7] can be used which predicts bunch lengths for an impedance which follows a particular power law.

The very successful theory of coupled bunch oscillations [8] does not predict instabilities of the single bunch - except for the Robinson instability, which can be easily cured by proper tuning. If coupling of different bunch shape modes is included [9] one can indeed obtain a stability criterion which agrees with the local coasting beam criterion over a certain range, but it requires the presence of a single, strong resonant impedance at very high frequencies (often above cut-off of the vacuum pipe) which appears unlikely to exist in all accelerators.

By including negative mode numbers for the bunch shape oscillations - or rather by not excluding them, as they are naturally present in the Fourier spectrum of a coherently oscillating bunch - this situation changes drastically. Mathematically, the expression for the shift of the synchrotron frequency then becomes quadratic rather than linear, and leads to complex solutions (instability) above a certain threshold. Physically, this may be interpreted as a coupling of bunch shape oscillations with a positive mode number to those with the same, but negative mode number, and it leads to a complete loss of focusing of the coherent motion. The bunch will then lengthen and widen incoherently until focusing is re-established, thereby leading to turbulent signals which persevere in electron machines, where a dynamical equilibrium between radiation damping and the blow-up forces is established.

The impedance required to drive this mechanism is at much lower frequencies, and in particular could be the fundamental RF cavity resonance which explains its presence in all bunched beam machines. The very large frequency shifts predicted by this model may be unobservable because of Landau damping by the frequency spread in the bunches.

Coherent bunch oscillations

The signal induced at position θ by a particle with charge e circulating in a storage ring with revolution frequency ω_{o} is given by a superposition of delta functions

$$s(\theta,t) = e \sum_{k=-\infty}^{\infty} \delta(t-\tau - \frac{\theta}{\omega_0} - \frac{2\pi k}{\omega_0})$$
(1)

where τ is the time delay of the particle with reference to the synchronous one. In the absence of perturbation, it is given by the synchrotron motion

$$\tau = \hat{\tau} \cdot \cos(\omega_{\rm s} t + \psi_{\rm o}) \tag{2}$$

where ω_s is the synchrotron frequency, $\hat{\tau}$ the oscillation amplitude and ψ_0 the initial phase angle. The signal can be Fourier analyzed [10] and one obtains

$$s(\theta, t) = \frac{\omega_{o}e}{2\pi} \sum_{m,p}^{\infty} J_{m}(p\omega_{o}\hat{\tau}) \cdot exp\left[j\left(\omega_{mp}t - p\theta + m(\psi_{o} - \frac{\pi}{2})\right)\right](3)$$

where the spectral frequencies are given by

$$\omega_{\rm mp} = p\omega_{\rm o} + m\omega_{\rm s} \tag{4}$$

We emphasize that both summation indices m and p run from $-\infty$ to $+\infty$, and negative values of m are just as valid as positive ones.

If we sum this signal over all particles with the same amplitude, it will in general average to zero unless the initial phases ψ_{0k} differ by a multiple of $2\pi/m$. Only then the terms $\exp(jm\psi_{0k})$ will add in phase and give a coherent signal. We also integrate over all amplitudes taking into account the distribution $\rho\left(\hat{\tau}\right)$ to get for the signal of a bunch oscillating at the m-th mode

$$S_{m}(\theta,t) = I_{b} \cdot \sum_{p=-\infty}^{\infty} exp \left[j(\omega_{mp}t - p\theta - \frac{m\pi}{2}) \right] \cdot \int_{0}^{\tau_{max}} J_{m}(p\omega_{o}r)\rho(r)dr$$
(5)

where I_b is the bunch current. For negative values of the mode number m, we can use the relation

$$J_{m}(x) = e^{\int \pi m} J_{m}(x)$$
 (6)

to obtain the same expression with ω_{mp} replaced by $\omega_{-m,\,p}\left(m\!\!>\!\!0\right)$.

The signal strength at the negative satellite frequencies $\omega_{-m,\,p}$ = $p\omega_{0}-m\omega_{s}$ is thus exactly the same as at the positive satellites, but the impedances will in general be different.

Coherent frequency shift

For vanishing beam current, the frequency ω_m of the m-th coherent oscillation mode of a bunch is just m times the (incoherent) synchrotron frequency of the single particles. For larger currents, the coherent frequency changes. For a parabolic bunch, the coherent frequencies can be found from the zeros of an infinite determinant

$$\left|\omega_{\rm m} - m\omega_{\rm s} - M_{\rm mk}\right| = 0 \tag{7}$$

where the matrix elements ${\ensuremath{M_{mk}}}$ are given by

$$M_{mk} = j \frac{32}{\pi} \frac{m\omega_s}{q+1} \frac{I_b}{B^3 h V \cos \psi_s} \left(\frac{Z}{n}\right)_{eff}^{m,k}$$
(8)

Here I_b is the current per bunch, $B=\omega_o\tau~/2\pi$ the bunching factor (τ - full bunch length in time), h the harmonic number, V the voltage seen by the beam, ψ_s the stable phase angle ($\cos\psi_s<\emptyset$ above transition), and the effective impedance is defined by

$$\left(\frac{z}{n}\right)_{\text{eff}}^{m,k} = \frac{\sum_{p} z(\omega_{mp})h_{mk}(p)/p}{\sum_{p} h_{mm}(p)}$$
(9)

(10)

are the (cross) power densities of the oscillation modes with line densities λ_m (t), resp. their Fourier transforms $\widetilde{\lambda}_m\left(\omega\right)$.

 $h_{mk}(p) = \overset{\circ}{\lambda}_{m}(\omega_{mp}) \overset{\circ}{\lambda}_{k}(\omega_{mp})$

The radial mode number $q = |m| + 2\ell$, $\ell = 0, 1, 2$. expresses the fact that there is an infinity of higher radial modes for each value of m. However, the lowest radial mode with q = |m| has always the strongest excitation.

If we neglect mode coupling ($M_{mk} = \emptyset$ except for m = k), the matrix in eq. (7) becomes diagonal, and we simply get

$$\omega_{\rm m} = m\omega_{\rm s} + M_{\rm mm} \tag{11}$$

If we include negative mode numbers, however, the matrix becomes bi-diagonal and we find (m > 0)

$$\begin{vmatrix} \omega^{+m\omega} s^{-M} - m, -m & M_{-m, +m} \\ M_{-m, -m} & \omega^{-m\omega} s^{-M} mm \end{vmatrix} = 0$$
(12)

Since M_{mk} has a factor m but not k (see eq. 8), we see that $M_{-m,m} = M_{-m,-m} = -M_{mm}$ while $M_{m,-m} = M_{mm}$ and hence we obtain the quadratic equation for ω_m

$$\omega_{\rm m}^2 = {\rm m}^2 \omega_{\rm s}^2 + 2 {\rm m} \omega_{\rm s} {\rm M}_{\rm mm}$$
(13)

For $|M_{rum}| << m\omega_s$, we obtain approximately eq. (11), i.e. the coherent frequency shift $\Delta \omega_m = \omega_m - m\omega_s$ is given by the matrix element M_{mm} . However, the quadratic expression yields

$$\Delta \omega_{\rm m} = {\rm m}\omega_{\rm s} \left[\sqrt{1 + \frac{2{\rm M}_{\rm mm}}{{\rm m}\omega_{\rm s}}} - 1 \right]$$
(14)

We see now that even a purely reactive effective impedance (real $M_{\rm mm}$) can lead to an imaginary frequency shift, and hence instability, if

$$\operatorname{Re} M_{mm} < -\frac{m\omega_{s}}{2}$$
 (15)

The negative sign shows that the effective impedance must be capacitive, such as is always the case e.g. for the fundamental RF cavities since the bunches must be shorter than the RF wavelength, and hence sample predominantly the capacitive region above resonance.

Turbulent bunch lengthening

In Eq. (8), the voltage seen by the beam is the applied voltage $V_{\rm RF}$ reduced by the potential well deformation due to space charge. This can be expressed by [9]

$$v = v_{\rm RF} \left(\frac{\omega_{\rm s}}{\omega_{\rm so}}\right)^2$$
(16)

where $\omega_{_{\rm SO}}$ is the synchrotron frequency for vanishing current. We define for short the parameter

$$C_{\rm m} = j \frac{32}{\pi^4} \frac{I_{\rm b}}{B_0^{3} h \, V \cos \psi_{\rm s}} \left(\frac{Z}{n} \right)_{\rm eff}^{\rm m,m}$$
(17)

where $B_o = \ell_o/2\pi R$ (ℓ_o natural bunch length). One can generalize the expression for the incoherent tune shift [13] by replacing |Z/n| with the effective impedance for the stationary distribution $m = \emptyset$. With $x = \ell/\ell_o$ one finds then

$$\left(\frac{\omega_{\rm s}}{\omega_{\rm so}}\right)^2 = 1 - C_{\rm o}/x^3$$
 (18)

Below the onset of turbulence, ω_S/ω_{SO} is related to the bunch lengthening x by

$$\frac{\omega_{\rm s}}{\omega_{\rm so}} = {\rm x}^{-\kappa}$$
(19)

where κ = 1 for electrons and κ = 2 for protons. We can rewrite Eq. (13) as

$$\left(\frac{\omega_{\rm m}}{{\rm m}\omega_{\rm so}}\right)^2 = 1 - \frac{1}{{\rm x}^3} \left[{\rm C}_{\rm O} - \frac{2}{{\rm q}+1} {\rm C}_{\rm m} \right]$$
(20)

For the lowest dipole mode m=q=1, this becomes simply

$$\left(\frac{\omega_1}{\omega_{so}}\right)^2 = 1 - \frac{1}{x^3} \left[C_0 - C_1\right]$$
(21)

For long bunches the effective impedance, and hence Cm, become independent of the mode number m [12] and we find $\omega_1 = \omega_{\rm SO}$, i.e. the dipole frequency equals the synchrotron frequency for zero current.

For short bunches (compared to the resonant wavelength) the effective impedance of a resonator with shunt impedance R_s , quality factor Q and resonant frequency ω_r is approximated by [12]

$$\left(\frac{Z}{n}\right)_{\text{eff}}^{\text{m,m}} = j \frac{\omega_{\text{r}}}{\omega_{\text{o}}} \frac{R_{\text{s}}}{Q} \frac{B_{\text{o}}^{2}}{1-2m} \left(\frac{\pi x}{2}\right)^{2}$$
(22)

It is inductive for m=0 and capacitive for m>1. Hence $C_o-C_1 > 0$ above transition, and $\omega_1 < \omega_s < \omega_{so}$. The lowest dipole frequency becomes imaginary when $C_o-C_1 > x^3$, and an instability will occur. The threshold is given by

$$I_{\text{thresh}}^{m=q=1} = \frac{hV_{RF}(-\cos\psi_s)B^3}{Im\left(\frac{z}{n}\right)_{eff}^{m=0} - \left(\frac{z}{n}\right)_{eff}^{m=1}}$$
(23)

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Since for short bunches $C_1 = -C_0$ (see Eq. 22), we find at the threshold $\omega_s = \omega_{so}/2$ and $V = V_{RF}/2$. The corresponding potential well bunch lengthening is found from Eq. 19, and is $\sqrt{2}$ for electrons and $4\sqrt{2}$ for protons. *An improved version uses the m=1 mode.

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In particular, one impedance for which the bunches are always shorter than the resonant wave length is the fundamental of the RF cavities. Then $\omega_r/\omega_o = h$ and

$$I_{\text{thresh}}^{m=q=1} = \frac{\pi^2}{16} \frac{V_{RF}^{\cos\psi}s}{R_s/Q} B$$
 (24)

for very short bunches. For longer bunches, the difference between the effective impedances with m=0 and m=1 becomes independent of bunch-length, and the threshold current proportional to B^3 . In addition to the fundamental RF resonance, a large number of resonances due to vacuum chamber cross-section variations and higher modes in the RF cavities are often described by a broad band impedance which peaks at the vacuum chamber cutoff frequency. Such an impedance would cause a threshold only if the bunch frequency were higher than the cut-off frequency. However, at high frequencies, the real part of the impedance decreases slower than that of a broad band resonator, and the capacitive part is thereby enhanced. This can lead to a threshold even when the bunch frequency is lower than the cut-off frequency of the vacuum chamber.

For Hermitian or sinusoidal modes, the infinite series for the effective impedance of a resonator can be summed analytically [12] and the exact expression for the threshold - although still complicated - is readily evaluated by computer.

Above the threshold, the bunch must blow up until equilibrium is re-established. Since the coherent mode frequency is reduced to zero, focusing of that mode is completely lost. The bunch will both lengthen and widen, giving rise to a large number of higher mode signals known as turbulence. For electrons, a dynamic equilibrium with radiation damping will cause continuous activity. For protons, there will be an overshoot if the threshold is exceeded suddenly - e.g. by injection into a different machine - while the bunch should stay at the limit of stability if the parameters are changed adiabatically.

The equilibrium bunch length will no longer be given by Eq. (19) but by the condition $\omega_m = \emptyset$ or

$$\left(\frac{\ell}{2\pi R}\right)^{3} = \frac{16}{\pi^{5}} \xi \operatorname{Im}\left[\left(\frac{Z}{n}\right)_{\text{eff}}^{\text{m=0}} - \frac{2}{q+1}\left(\frac{Z}{n}\right)_{\text{eff}}^{\text{m,m}}\right]$$
(25)

where

 $\xi = -\frac{2\pi I_b}{hV \cos \psi_s} = \frac{\alpha I_b}{\nu_{so}^2 E/e}$

is the "scaling parameter" introduced in ref. 7. Since the effective impedance is a function of bunch length, Eq. (25) is a transcendental equation. The numerical solution yields the bunch lengthening ℓ/ℓ_0 shown in Fig. 1 for a particular set of parameters (LEP), as well as the incoherent and coherent synchrotron frequencies ω_s/ω_{so} and ω_1/ω_{so} which are obtained from Eqs. (18) and (20). The coherent frequency shift is very large, which appears to contradict observation. However, since the coherent frequencies are shifted below the incoherent frequency of the beam center, they are in the region of the incoherent frequencies of particles with finite synchrotron amplitude. We expect therefore that Landau damping should make their observation difficult [14]. On the other hand, the rigid dipole oscillation - which shifts the potential well without deformation - is actually not included in this analysis. If excited by an external modulation, the bunch will thus show no frequency shift, resp. a very small one due to the non-linearities of the RF potential.



Fig. 1 Effect of the fundamental resonance of the RF cavities on bunch length and synchrotron frequencies. $p = (16/\pi^5)(\xi/B_0^3) \vee I_h$

Conclusions

We present an analysis of single-bunch oscillations which leads to instability above a certain threshold current. It is essentially caused by the coupling of positive and negative mode numbers of the same bunchshape oscillation. It can be driven by resonant impedances such as the fundamental mode of the RF cavities or the broad band impedance due to a larger number of resonances in the RF cavities and the vacuum chamber. These exist in every storage ring or accelerator, and no unphysical impedances are thus required to provide coupling.

References

- [1] A. Lebedev, Physics with Intersecting Storage Rings Academic Press 1971.
- [2] A.M. Sessler, PEP-Note 28 (1973).
- [3] P. Channel, Thesis Univ. California (1975).
- [4] P. Channel, A.M. Sessler, N.I.M. 136 (1976) 473.
- [5] M. Months, E. Messerschmid, N.I.M. 141 (1977) 1, 144 (1977) 119.
- [6] D. Boussard, CERN report LAB II/RF 75-2 (1975).
- [7] A. Chao, J. Gareyte, PEP-Note 224 (1975).
- [8] F. Sacherer, IEEE Trans. NS-20 (1973) 825.
 [9] F. Sacherer, IEEE Trans. NS-24 (1977) 1393.
- [10] J. Laclare, Saclay report LNS 021 (1978).
- [11] A. Bofmann, LEP notes 70/74 (1978) and 168 (1979).
 [12] B. Zotter, CERN report ISR-80-3 (1980).
- [13] S. Hansen et al., IEEE Trans. NS-22 (1975) 1381.
- [14] G. Besnier, Thesis Univ. Rennes (1978).