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COHERENT NORMAL MODES OF COLLIDING BEAMS*

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Abstract

The coherent normal modes of the nonlinear colliding beam system are calculated using averaging methods to find an equivalent infinite system of linear coupled oscillators. The condition of a consistent, stationary state leads to an eigenvalue equation for the characteristic frequencies and transverse exitation distributions of the modes. Results are presented for modes along the narrow axis of a Gaussian ribbon beam. The tune split is related to $L/\sqrt{(I^+I^-)}$ for different values of I^+/I^- and of σ^+/σ^-.

Method

Assumptions and Definitions

It will be assumed that the particles in each bunch have a betatron excitation in both transverse directions due to radiation, and that the equilibrium charge density is proportional to $\mathrm{Exp}(-x_x^2/2\sigma_x^2-x_z^2/2\sigma_z^2)$, where $\sigma_x \gg \sigma_z$. It is also assumed that there are no lattice resonances or other couplings except due to the collisions, that there are two collisions per revolution, and that the particle energy $\gamma \gg 1$.

A coherent mode is defined as a correlated oscillation of all particles in phase and amplitude about each particle's equilibrium betatron phase and amplitude. The oscillating displacement must be a stationary solution of the undriven colliding beam system. The only modes easily observable are ones with a large center of charge component in either bunch. Also only modes in the z direction will be considered here.

Parametrization of Tune Spread

The colliding beam system can be described as an equilibrium charge distribution in each bunch, and small oscillations about the equilibrium distribution. The equilibrium charge distribution generates an electric field E_z which is odd in x_z. This field produces a force which is quadrupole to leading order. If the tune shift due to this quadrupole is small compared to the fractional tune in the z direction, then the higher order multipoles of the field can be averaged over a betatron cycle to give an equivalent quadrupole force which is a function of the betatron amplitude in the z direction[1]. Therefore, each particle has a betatron frequency Ω_z depending on its betatron amplitude x_{oz}. The betatron frequencies now occupy a continuum. Since it is assumed that there are no important lattice resonances, the z and x direction betatron motions are uncorrelated. Then the multipole structure of E_x in x_x can be averaged independently of the multipole structure in the z direction. The result is that the z direction betatron frequency is $\Omega(x_{ox},x_{oz})$, where x_{ox},x_{oz} are the betatron amplitudes in the x and z directions.

Coherent Bunch Coupling

The small oscillations of the charge density generate an electric field δE_z which is even in x_z, and oscillates at some characteristic frequency ω. This field produces an oscillating dipole force to leading order, and the higher multipoles can be averaged in the same manner as the equilibrium field to give an equivalent oscillating dipole which is a function of x_{ox},x_{oz}.

The result of the averagings is that small coherent motion of the nonlinear system can be represented as an infinite set of coupled linear oscillators with a distribution of resonant frequencies corresponding to the particle betatron amplitudes. The equations of the equivalent linear system are linear integral equations where the coherent frequency appears as an eigenvalue. These equations can be solved by evaluation of the matrix elements of the integral operator in a basis of orthonormal functions, and solving the resulting matrix equation.

Solution

Single Particle Motion

Let x_{ox},x_{oz} be the equilibrium betatron amplitude of a given particle, and let χ_o be the amplitude of a small coherent oscillation. The instantaneous displacements are given by

$$x = x_o\cos\Omega t \;,\; \chi = \chi_o\cos\omega t \tag{1}$$

Also define dimensionless coordinates $a = x/\sqrt{2}\sigma$, $a_o = x_o/\sqrt{2}\sigma$. Henceforth, the two colliding bunches will be assumed to be e^+ and e^-, and a superscript will be used to indicate motion in a specific bunch. The single particle equation of motion is that of a driven harmonic oscillator with a small nonlinear perturbation

$$\ddot{x}_z^+ + \Omega_o^2 x_z^+ = -\varepsilon_1 f(x_x^+,x_z^+)x_z^+ + \varepsilon_2\delta x_o^-(x_x^+,x_z^+)\cos\omega t \tag{2}$$

The identical equation with + and - will also be assumed. f and δx_o are perturbing functions. f contains the nonlinear structure of the force due to the equilibrium charge distribution, normalized such that $f(0,0) = 1$, and δx_o^- is the effective displaced charge of the e^- bunch as seen by a particle in the e^+ bunch. δx_o^- depends on the integral of χ_o^- over all amplitudes. Ω_o is the single particle betatron frequency in the z direction.

Charge Density and Fields

The static charge density is

$$dQ(a_x,a_z) = eN\rho(a_x,a_z)da_x da_z \tag{3}$$

$$\rho(a_x,a_z) = \frac{1}{\pi}\mathrm{Exp}(-a_x^2-a_z^2)$$

The resulting equilibrium field in cgs units is[2]

$$E_z = \frac{2eN}{\sigma_x\sigma_z} f(a_x,a_z)x_z \tag{4}$$

$$f = \frac{\sqrt{\pi}}{2}\mathrm{Exp}(-a_x^2)\mathrm{Erf}(a_z)/a_z$$

The field due to coherent oscillation requires integrals over both the betatron amplitude and instantaneous displacement. The required charge distribution is

$$dQ = eN\rho(a_{ox},a_{oz},a_x,a_z)da_{ox}da_{oz}da_x da_z \tag{5}$$

$$\rho(a_{ox},a_{oz},a_x,a_z) = \frac{4a_{ox}a_{oz}\mathrm{Exp}(-a_{ox}^2-a_{oz}^2)}{\pi^2(a_{ox}^2-a_x^2)^{1/2}(a_{oz}^2-a_z^2)^{1/2}}$$

The z component of the field acting on a particle at x_x^+,x_z^+ due to a charge element at x_x^-,x_z^-, where dQ is defined in (5), is

$$dE_z^- = \frac{2dQ(a_{ox}^-,a_{oz}^-,a_x^-,a_z^-)(x_z^+-x_z^-)}{(x_z^+-x_z^-)^2+(x_x^+-x_x^-)^2} \tag{6}$$

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Define $d\delta E_z$ to be the change in field due to a displacement of dQ by χ.

$$d\delta E_z^- = \chi^-(a_{ox}^-, a_{oz}^-) \frac{\partial}{\partial x_z^+}(dE_z^-) \tag{7}$$

The differentiation is performed with respect to x_z^+ instead of x_z^- because the charge in dQ is constant, so ρ must not be differentiated. The derivative does not exist at the singularity of dE_z but this can be handled by doing the integral over a_x^- first, and using $\sigma_x \gg \sigma_z$

$$\delta E_z^- = \frac{2\pi e N}{\sigma_x \sigma_z} \iint_0^\infty da_{ox}^- da_{oz}^- \rho(a_{ox}^-, a_{oz}^-, a_x^+, a_z^+) \chi^-(a_{ox}^-, a_{oz}^-) \tag{8}$$

define

$$\delta x^-(a_x^+, a_z^+) = \pi \iint_0^\infty da_{ox}^- da_{oz}^- \rho(a_{ox}^-, a_{oz}^-, a_x^+, a_z^+) \chi^-(a_{ox}^-, a_{oz}^-) \tag{9}$$

$$\delta E_z^- = \frac{2eN}{\sigma_x \sigma_z} \delta x^- \tag{10}$$

Note that for χ constant, $\delta x = \chi$, and that (10) is consistent with (4).

Averaging

Using the method of Krylov and Bogoliubov[1], a solution is supposed for a given particle, of the form

$$x_x = x_{ox} \cos\psi_x \ , \ x_z = x_{oz} \cos\psi_z \tag{11}$$

Then expressions for $\dot{x}_{oz}$, $\dot{\psi}_z$ can be found

$$\dot{x}_{oz}^+ = -\frac{\varepsilon_1}{\Omega_o} \sin\psi_z^+ f(a_x^+, a_z^+) x_{oz}^+ \cos\psi_z^+ + \frac{\varepsilon_2}{\Omega_o} \sin\psi_z^+ \delta x_o^-(a_x^+, a_z^+) \cos\omega t \tag{12}$$

$$\dot{\psi}_z^+ = -\frac{\varepsilon_1}{\Omega_o} \cos\psi_z^+ f(a_x^+, a_z^+) \cdot \cos\psi_z^+ + \frac{\varepsilon_2}{x_{oz}^+ \Omega_o} \cos\psi_z^+ \delta x_o^-(a_x, a_z) \cos\omega t + \Omega_o$$

Averaging over one cycle of ψ

$$\langle \dot{x}_{oz}^+ \rangle = -\frac{\varepsilon_2}{\Omega_o}(2\ )^{-2} \iint_0^{2\pi} \sin^2\psi_z \ \delta x_o^-(a_x^+, a_z^+) d\psi_x d\psi_z \ \sin\Delta\omega t \tag{13}$$

$$\langle \dot{\psi}^+ \rangle = \Omega_o + \frac{\varepsilon_1}{\Omega_o}(2\pi)^{-2} \iint_0^{2\pi} \cos^2\psi_z \ f(a_x^+, a_z^+) d\psi_x d\psi_z - \frac{\varepsilon_2}{x_{oz}^+ \Omega_o}(2\pi)^{-2} \iint_0^{2\pi} \cos^2\psi_z \ \delta x_o^-(a_x^+, a_z^+) d\psi_x d\psi_z \ \cos\Delta\omega t$$

where $\Delta\omega = \omega - \Omega$ is the beat frequency between the averaged equilibrium betatron frequency Ω and the coherent frequency ω, and $\Omega = \Omega_o + \Delta\Omega(a_{ox}, a_{oz})$.

$$\Delta\Omega(a_{ox}, a_{oz}) = \frac{\varepsilon_1}{\Omega_o} \iint_0^{2\pi} d\psi_x d\psi_z \cos\ _z f(a_x, a_z)\ /(2\pi) \tag{14}$$

$$= \Delta\Omega(0,0)\ F_x(a_{ox})\ F_z(a_{oz})$$

where f is defined by (4) and (11), and

$$\Delta\Omega(0,0) = \frac{\varepsilon_1}{2\Omega_o} \tag{15}$$

Integrating (14)[3]

$$F_x(a_{ox}) = \mathrm{Exp}\left(-\frac{a_{ox}^2}{2}\right) I_o\left(\frac{a_{ox}^2}{2}\right) \tag{16}$$

$$F_z(a_{oz}) = \mathrm{Exp}\left(-\frac{a_{oz}^2}{2}\right) \left(I_o\left(\frac{a_{oz}^2}{2}\right) + I_1\left(\frac{a_{oz}^2}{2}\right)\right)$$

where I_o, I_1 are modified bessel functions.

Coherent Motion

Note from (13) that $\langle \dot{x}_{oz} \rangle$ and $\langle \dot{\psi}_z \rangle$ oscillate about equilibrium with frequency $\Delta\omega$, and that the peak phase displacement is not proportional to the peak amplitude displacement as in a linear oscillator. Define u and v to be displacements of amplitude and phase from the equilibrium values x_{oz} and ψ_z

$$u(a_{ox}, a_{oz}) = \int \langle \dot{x}_{oz} \rangle dt = u_o \cos\Delta\omega t \tag{17}$$

$$v(a_{ox}, a_{oz}) = \int \langle \dot{\psi}_z \rangle dt = -v_o \sin\Delta\omega t$$

u and v contribute to χ according to their projections on the spatial axis of phase space

$$\chi(a_{ox}, a_{oz}) = u_o \cos\Omega t \cos\Delta\omega t - v_o \sin\Omega t \sin\Delta\omega t$$

discarding terms that will average to zero over Ωt

$$\chi(a_{ox}, a_{oz}) = \cos\omega t\ (u_o \cos^2\Omega t + v_o \sin^2\Omega t\ ) \tag{18}$$

$$\chi_o(a_{ox}, a_{oz}) = \frac{1}{a_{oz}^2}(a_z^2\ u_o + (a_{oz}^2 - a_z^2)\ v_o\ )$$

Define

$$\chi_o^+ = \begin{pmatrix} u_o^+ \\ v_o^+ \end{pmatrix} , \ \chi_o = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix} \tag{19}$$

To evaluate ε_1 and ε_2, consider the case where a_{ox} and a_{oz} are small so that $f(a_x, a_z) = 1$, and also $\Delta\Omega(a_{ox}, a_{oz}) = \Delta\Omega(0,0)$, $\delta x_o(a_x, a_z) = \chi_o(0,0)$. Let $X = x + \chi$ be a complete solution. Using (1),

$$\ddot{X} + \Omega_o^2 X = (\Omega_o^2 - \Omega^2)x + (\Omega_o^2 - \omega^2)\chi \simeq -2\Omega_o(\Delta\Omega x + \Delta\omega\chi) \tag{20}$$

For a linear quadrupole, the shift in betatron frequency is, assuming the equilibrium density ρ

$$\Delta\Omega(0,0) = \frac{\beta_z^* r_e N n}{\gamma T_o \sigma_x \sigma_z} \tag{21}$$

where n = 2 is the number of collisions per revolution, T_o is the revolution period, r_e is the classical radius of the electron, β_z^* is the beta function at collision, and γ is the energy. From (15) and (21),

$$\varepsilon_1 = \frac{2\Omega_o \beta_z^* r_e N n}{\gamma T_o \sigma_x \sigma_z} \tag{22}$$

For a dipole shaking field, the shaking response is

$$\chi_o^+ = \frac{\beta_z^* r_e N n}{\gamma T_o \sigma_x \sigma_z \Delta\omega} \delta x_o^-(0,0) \tag{23}$$

From (2) and (20), keeping only terms in χ,

$$2\Omega_o \Delta\omega \chi_o^+ = \varepsilon_2 \delta x_o^-(0,0) \tag{24}$$

$$\varepsilon_2 = \varepsilon_1 = 2\Omega_o \Delta\Omega(0,0)$$

Combining equations (5),(9),(13),(17),(18), and (19), and changing variables from ψ_z to a_z, a system of integral equations is obtained.

$$\Delta\omega\ \chi_o^+(a_{ox}^+,a_{oz}^+) = \frac{\pi\varepsilon_2}{\Omega_o} \iint_0^\infty da_{ox}^- da_{oz}^- A(a_{ox}^+,a_{ox}^-) B(a_{oz}^+,a_{oz}^-) \cdot \rho(a_{ox}^-)\rho(a_{oz}^-)\chi_o^-(a_{ox}^-,a_{oz}^-) \tag{25}$$

$$\Delta\omega = \omega - \Omega_o - \Delta\Omega(a_{ox},a_{oz})$$

$$\rho(a_o) = 2a_o\ \mathrm{Exp}(-a_o^2)$$

$$A(a_o,b_o) = \frac{2}{\pi^2} \int_0^{\mathrm{Min}(a_o,b_o)} da\ (a_o^2 - a^2)^{-\frac{1}{2}}(b_o^2 - a^2)^{-\frac{1}{2}}$$

$$B(a_o,b_o) = \frac{2}{\pi^2} \int_0^{\mathrm{Min}(a_o,b_o)} da\ (a_o^2 - a^2)^{-\frac{1}{2}}(b_o^2 - a^2)^{-\frac{1}{2}}(a_o b_o)^{-2}. \begin{bmatrix} a^2(a_o^2-a^2) & (a_o^2-a^2)(b_o^2-a^2) \\ a^4 & a^2(b_o^2-a^2) \end{bmatrix}$$

A and B can be evaluated analytically in complete elliptic integrals. Define

$$\lambda = (\omega - \Omega_o)/\Delta\Omega(0,0) \tag{26}$$

$$\xi_x = F_x(a_{ox})\ ,\ \xi_z = F_z(a_{oz})$$

$$\Delta\Omega(a_{ox},a_{oz}) = \Delta\Omega(0,0)\xi_x\xi_z$$

changing variables in (25)

$$\lambda\ \chi^+(\xi_x^+,\xi_z^+) = \xi_x^+\xi_z^+\ \chi^+(\xi_x^+,\xi_z^+)\ -2\pi \iint_0^1 d\xi_x^- d\xi_z^- K^- \chi^-(\xi_x^-,\xi_z^-) \tag{27}$$

Expand in an orthonormal basis. A suitable basis is a product of legendre polynomials on ξ_x,ξ_z .

$$\Psi_N(\Gamma) = \sqrt{2m+1}\ \sqrt{2n+1}\ P_m(2\xi_x-1)\ P_n(2\xi_z-1)$$

$$\chi_N^+ = \int d\Gamma\, \Psi_N\, \chi^+$$

$$K_{MN}^- = \iint d\Gamma^+ d\Gamma^- K^-(\Gamma^+,\Gamma^-)\, \Psi_M(\Gamma^+)\, \Psi_N(\Gamma^-)$$

$$\Xi_{MN} = \int d\Gamma\ \xi_x\xi_z\, \Psi_M(\Gamma)\, \Psi_N(\Gamma)$$

so equation (27) becomes

$$\lambda\ \chi_M = \begin{bmatrix} \Xi_{MN}^+ & -2\pi K_{MN}^- \\ -2\pi K_{MN}^+ & \Xi_{MN}^- \end{bmatrix} \chi_N \tag{28}$$

where $\Gamma = (\xi_x,\xi_z)$, $d\Gamma = d\xi_x d\xi_z$, and the integrals over $d\Gamma$ are over the unit square.

The beam-beam transfer function may also be calculated by adding a driving vector to the right hand side of (28) ,and setting ω equal to the driving frequency. Then $\chi(\omega)$ can be obtained for a given shaking configuration.

Results

The matrix system was truncated to 60 × 60 for numerical solution. This involves severely truncating the expansion in each degree of freedom. The expansion in ξ_z was truncated to five dimensions, and the expansion in ξ_x was truncated to three dimensions. Variations of the truncation are used to estimate the error induced by the truncation. A mode with odd symmetry between electrons and positrons, and very little coupling to the truncated degrees of freedom appears at λ = 1.34 with equal bunch sizes and currents. Additional modes with moderate coupling to truncated degrees of freedom and a significant center of charge motion appear at λ = 0.79 with odd symmetry, and at λ = 0.63 with odd symmetry. Also, a collection of modes with even symmetry appear close to λ = 0.095 .

Since the eigenvalues of 1.34 and 0.095 seem to correspond to lines observable in CESR, the dependence of these on asymmetry between the bunches was calculated. Define

$$p = (I^+/I^-)^{\frac{1}{2}},\quad q = (\sigma_z^+/\sigma_z^-)^{\frac{1}{2}}$$

Then p was varied with q = 1 , and q was varied with p = 1 . The results are summarized below.

p (q=1)	λ_{even}	λ_{odd}
1.0	0.097	1.340
1.1	0.096	1.349
1.2	0.095	1.375
1.4	0.088	1.460
1.6	0.085	1.573
1.8	0.077	1.704
2.0	0.075	1.846

q (p=1)	λ_{even}	λ_{odd}
1.0	0.097	1.340
1.1	0.096	1.347
1.2	0.095	1.368
1.4	0.100	1.438
1.6	0.110	1.536
1.8	0.120	1.650
2.0	-	1.772

The matrix with q = 2.0 had no eigenvalue corresponding to a strong even mode with λ near zero. This is probably due to the matrix truncation effects, which worsen with q.

For Gaussian, ribbon shaped beams, the luminosity is proportional to $\Delta\Omega(0,0)$ for motion in the thin direction of the beam. Using (21),

$$L = \frac{T_o I^2}{4\pi e^2 \sigma_x \sigma_z} = \frac{T_o \gamma}{8\pi e r_e \beta_z^*}\ \Delta\Omega(0,0)\ I \tag{29}$$

Therefore, observed tune splits should be linear in L/I.

References

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