IEEE Transactions on Nuclear Science, Vol. NS-28, No. 3, June 1981

VERTICAL FAST BLOW-UP IN A SINGLE BUNCH*

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Introduction

In this paper we study vertical coherent instabilities in bunched beams with an emphasis on single bunch instabilities in which the growth time is less than the period of synchrotron oscillations. Single bunch instabilities have been studied by many people,[1-3] however, in most treatments the standard assumption is that the coherent force can be treated as a perturbation compared to the synchrotron force. This simplifies the problem greatly since in this way an individual synchrotron mode is decoupled (or coupled only to a neighboring mode). In the regime of fast blow-up (growth time < synchrotron period) this assumption is not valid, and it is therefore necessary to include the coupling of all the synchrotron modes.

In addition, since the beam is bunched, all the revolution modes are coupled. This is true because the perturbed distribution of particles, ψ_1, must be bunched azimuthally at least as much as the unperturbed bunch distribution, ψ_o. So the solution must lead to a wave packet rather than a plane wave. This introduces a fundamental complication into the problem in that we must solve an integral equation rather than a dispersion relation.

To study the problem quantitatively:

1. We introduce the Vlasov equation with single particle equations of motion, and reduce this to an integral equation.
2. We restrict ourselves to the following regime.
   - A. $\mathrm{Im}(\omega) > \omega_s$.
   - B. Frequency $\sim \omega_c$, the cutoff frequency.
   - C. Broad band impedance + space charge.
   - D. Width $Z_\perp(\omega)$ > 1/bunch length.
3. We solve the integral equation in this regime analytically.
4. We apply the results to a calculation of the stability of the injected bunch at ISABELLE.

The Equations of Motion

We are interested in vertical coherent instabilities. Since the coherent forces are typically much less than the focusing forces, we look for growth rates which are much less than the betatron frequency. Consistent with this assumption, we replace the true betatron motion by harmonic oscillations. In order to include the effect of synchrotron oscillations, we linearize this motion. With these assumptions the single particle equations of motion for our problem become

$$\ddot{\tau} + \omega_s^2 \tau + \ldots = 0 \tag{1}$$

$$\ddot{y} + \omega_y^2 (1 + 2 a\dot{\tau}) y = \frac{F(\tau,t)}{\gamma m} \tag{2}$$

$$\tau = \phi/\omega_o = \frac{\theta - \omega_o t}{\omega_o}, \quad \theta = \text{azimuth}, \quad \dot{\tau} = \frac{\Delta\omega_o}{\omega_o} = \eta \frac{\Delta p}{p} \tag{3}$$

$$a = (1 - \xi/\eta), \quad \eta = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}, \quad \omega_y = \nu\omega_o, \quad \xi = \frac{\Delta\nu}{\nu}\Big/\frac{\Delta p}{p} \tag{4}$$

ω_o, ω_y, and ω_s are the revolution frequency, betatron frequency, and synchrotron frequency respectively, and $F(\tau,t)$ is the vertical coherent force. The first equation gives the synchrotron motion while the second gives the coherently forced betatron oscillation.

The coherent motion of a bunch of particles is given by the Vlasov equation. This can be written

$$\frac{\partial\psi}{\partial t} + [\psi, H] = 0 \tag{5}$$

where ψ is the distribution function in phase space, [,] is the Poisson bracket, and H is the hamiltonian which yields equations (1) and (2). In particular we are interested in stability, so we linearize the above equation by the substitution

$$\psi = \psi_o + \psi_1 e^{-i\omega t} \tag{6}$$

which yields

$$-i\omega\psi_1 + [\psi_1, H_o] + [\psi_o, H_c] = 0 \tag{7}$$

where H_o is the single particle hamiltonian, H_c is the coherent hamiltonian, $H_c = -y\,F(\tau,t)$, and $[\psi_o, H_o] = 0$. The coherent force in Eq.(2) can be written with a transverse impedance.

$$\frac{F(\tau,t)}{\gamma m} = \int_{-\infty}^{\infty} dk\; G(\omega + k\omega_o) D(k) e^{i(k\omega_o\tau - \omega t)} \tag{8}$$

where
$$G(\omega) = i e^2 \omega_o^2 Z_\perp(\omega)/(2\pi m_o \gamma c) \tag{9}$$

and
$$D(k) = 1/2\pi \int_{-\infty}^{\infty} e^{-ik\omega_o\tau} y\psi_1(y,p_y,\tau,p_\tau)\, dy\, dp_y\, d\tau\, dp_\tau. \tag{10}$$

This force is valid for a short wake field (less than one turn). If we have a long wake field, then the periodicity in τ becomes important and we find a sum (Fourier series) rather than an integral.

Before substituting into the Vlasov equation it is convenient to introduce action angle variables:

$$y = \sqrt{2J_y/\omega_y}\cos\theta_y, \quad \tau = \sqrt{2J_s/\omega_s c^2}\cos\theta_s$$
$$p_y = -\sqrt{2J_y\omega_y}\sin\theta_y, \quad p_\tau = -\sqrt{2J_s\omega_s c^2}\sin\theta_s. \tag{11}$$

If we let $\tilde{\tau}(J_s) = \sqrt{2J_s/\omega_s c^2}$ and use equation (7), we find the Vlasov equation for our problem

$$-i\omega\psi_1 + \omega_y(1 - a\omega_s\tilde{\tau}\sin\theta_s)\frac{\partial\psi_1}{\partial\theta_y} + \omega_s\frac{\partial\psi_1}{\partial\theta_s}$$
$$-\frac{\partial\psi_o}{\partial J_y}\sqrt{2J_y/\omega_y}\,\sin\theta_y\,\frac{F(\tilde{\tau}\cos\theta_s)}{\gamma m} = 0 \tag{12}$$

where we have again linearized the synchrotron oscillation. Our task now is to solve this equation; to find the eigenvalue ω, and eigenfunction ψ_1. If <u>any</u> solution has $\mathrm{Im}(\omega) > 0$, then the system is unstable. The growth rate in this circumstance is just $\mathrm{Im}(\omega)$. Recall that $[\psi_o, H_o] = 0$; this means that $\psi_o = \psi_o(J_s, J_y)$. In the coasting beam case $\psi_o = \psi_o(p_\tau, J_y)$ since there is no longitudinal focusing. It is this difference that leads to the coupling of different revolution modes.

We expect an integral equation due to the form of the force so we seek an equation for D(k) of the form

$$\bar{D}(k) = \int_{-\infty}^{\infty} dk' A(k,k')\bar{D}(k'). \tag{13}$$

Since equation (12) is a linear integro differential equation, such an integral equation should exist. In the coasting beam limit this must reduce to a dispersion relation of the standard form. Let us search for a solution in the neighborhood of $\omega \sim -\omega_y$, i.e. $\psi = \mathrm{Re}^{-i\theta_y}$. If we define

$$f = \int\int y\,\psi_1 dy\,dp_y = \pi\int_0^\infty \sqrt{2J_y/\omega_y}\,R\,dJ_y \tag{14}$$

then from Eq. (10) we have

$$D(k) = 1/2\pi \int_0^\infty f\, e^{-ik\omega_o\tau} d\tau\, dp_\tau. \tag{15}$$

This form for D(k) suggests that we multiply Eq. (12) by $\pi\sqrt{2J_y/\omega_y}$ and integrate over J_y. This yields

$$\left\{-i(\lambda - a\omega_y\tilde{\tau}\sin\theta) + \frac{\partial}{\partial\theta}\right\} f = \frac{-i\rho_o F(\tilde{\tau}\cos\theta)}{2\omega_y\omega_s} \tag{16}$$

*Work performed under the auspices of the U.S. Department of Energy.

where $\lambda \equiv (\omega + \omega_y)/\omega_s$ and we have dropped the s subscript. At this stage one could introduce synchrotron harmonics; however, since we are interested in shifts in frequency which might be large compared to the synchrotron frequency, we will take a different approach. It is easy to see that the periodic solution to Eq.(16) is:

$$f = \frac{-ie^{i\lambda\theta+ia\omega_y\tilde{\tau}\cos\theta}\rho_o}{(e^{-2\pi i\lambda}-1)} \int_\theta^{\theta+2\pi} \frac{F(\tilde{\tau}\cos\theta')}{2\omega_y\omega_s} e^{-i\lambda\theta'-ia\omega_y\tilde{\tau}\cos\theta'}d\theta' \tag{17}$$

The right hand side of Eq. (17) is a linear functional of D(k). So it is straightforward to show with Eq. (15) and (17) that after some simplification the kernal in Eq. (13) becomes

$$A(p,p') = \frac{G(\omega+p'\omega_o+a\omega_y)}{4\pi\omega_y\omega_s} \times \tag{18}$$

$$\frac{-i}{(1-e^{2\pi i\lambda})} \int_0^{2\pi} e^{i\lambda t} \int_0^\infty \tilde{\tau}d\tilde{\tau}\, \rho_o(\tilde{\tau})\, J_o[K_{pp'}(t)\omega_o\tilde{\tau}]$$

where $$K_{pp'}(t) = [(p-p')^2-2pp(\cos t-1)]^{1/2} \tag{19}$$

$$\bar{D}(p) = D(p + a\nu), \text{ and } \int_0^\infty \tilde{\tau}d\tilde{\tau}\rho_o(\tilde{\tau}) = N. \tag{20}$$

The above result follows exactly from the Vlasov equation in Eq. (12). The problem has been reduced to the solution of an integral equation with eigenvalue equal to unity. The eigenvalue of Eq. (13) will have λ as a free parameter, so that after the integral equation is diagonalized we are left with an equation of the form $1 = E(\lambda)$, where $E(\lambda)$ is the eigenvalue. Thus, after diagonalization, we obtain an equation of the same form as the normal dispersion relation in coasting beam theory. In order to gain more insight into the integral equation, let us now specialize to a gaussian distribution for ρ_o.

Gaussian Bunch, High Frequency Fast Blow Up

We select a distribution function given by

$$\rho_o = N/\sigma^2\, e^{-\tilde{\tau}^2/2\sigma^2} = N/\sigma^2\, e^{-\tau^2/2\sigma^2} e^{-\dot{\tau}^2/2\omega_s^2\sigma^2} \tag{21}$$

which is gaussian in τ and $\dot{\tau}$. Physically this is gaussian in azimuth and in $\Delta p/p$. The last integral in Eq. (18) can be done easily with this distribution:

$$= Ne^{-(p-p')^2\sigma_o^2/2 + \sigma_o^2 pp'(\cos t-1)} \tag{22}$$

where $\sigma_o = \sigma\omega_o$ = rms spread in azimuth. If we set $\omega + a\omega_y \simeq -\omega_y\xi/\eta \equiv -\omega_\xi$, the kernal can be written

$$A(p,p') = N\, \frac{G(p'\omega_o - \omega_\xi)}{4\pi\omega_y\omega_s}\, e^{-(p-p')^2\sigma_o^2/2} \times B(pp',\lambda) \tag{23}$$

with $$B = \frac{-i}{1 - e^{2\pi i\lambda}} \int_0^{2\pi} e^{i\lambda t} e^{pp'\sigma_o^2(\cos t-1)} dt. \tag{24}$$

Notice that the mode coupling in the kernal is due mainly to the Fourier transform of ψ_o, as expected. The factor B plays the same role as the dispersion integral coasting beam theory; however, it also contains the development in terms of synchrotron harmonics. B can be written exactly in terms of a series with modified Bessel functions as

$$B = e^{-pp'\sigma_o^2} \sum_k \frac{I_k(pp'\sigma_o^2)}{\lambda-k}. \tag{25}$$

The usual approaches consider one term in this sum; however to study the regime of fast blowup ($\text{Im}(\omega) > \omega_s$ or $\text{Im}\,\lambda > 1$), we must, in principle, keep all of the terms, i.e. the entire integral B.

Now we would like to consider a broad band impedance at high frequency. This should lead to some simplification since this impedance is approximately constant over a broad range. Let Γ be the width of $G(\omega)$ and let G be peaked at the cutoff frequency, ω_c. We will consider the case when the bunch length is much longer than the wakefield, i.e.

$$\Gamma >> 1/\sigma,\ \omega_c >> 1/\sigma. \tag{26}$$

In the regime of fast blow up ($\text{Im}\lambda > 1$) it is easy to see that the last 2 factors in Eq. (23) are very small when p & p' are of opposite sign. This means that we need only consider the coupling of like frequencies. This is in contrast to the head-tail regime [one term in the sum in Eq. (25)] where opposite and like signs of p & p' contribute equally. If we now search for a solution to equation (13) in the neighborhood of the cutoff frequency, then the kernal in Eq. (23) is the product of a slowly varying function, $G(p'\omega_o-\omega_\xi) \times B(pp',\lambda)$, and a sharply spiked function, $\exp[-(p-p')^2\sigma_o^2/2]$, which we approximate by

$$A(p,p') \simeq \frac{NG(\omega_c)}{4\pi\omega_y\omega_s} e^{-(p-p')^2\sigma_o^2/2} \times B(p_o^2,\lambda) \tag{27}$$

with $p_o = (\omega_c + \omega_\xi)/\omega_o$. We should restrict the range of integration to be over the width of G; however, since $\Gamma >> 1/\sigma$, we can extend the range of integration with little error. In this case Eq. (13) can be solved by $\bar{D} = \exp(-ip\omega_o y)$ and we find

$$1 = \frac{NG(\omega_c)}{4\pi\omega_y\omega_s} B(p_o^2,\lambda)\, \frac{\sqrt{2\pi}}{\sigma_o}\, e^{-y^2/2\sigma^2}. \tag{28}$$

Since we are searching for instability, select the largest eigenvalue (y = 0); then using Eq. (9) we find

$$1 = ie\, \frac{I_{peak}Z_\perp(\omega_c)}{m_o\gamma c 4\pi\nu\omega_s} B(p_o^2,\lambda) \tag{29}$$

where $$I_{peak} = eN/(\sigma\sqrt{2\pi}). \tag{30}$$

We see that the solution of the integral equation has brought a factor of the local current (the peak current since we've used the largest eigenvalue). This makes sense physically because a broad band impedance corresponds to a very local interaction within the bunch.

We have solved the integral equation in this regime, and we are left with a relation which is formally identical to the normal coasting beam dispersion relation. In the next section we study this dispersion relation and apply these results to a calculation of the stability of the injected bunch at ISABELLE.

The Dispersion Relation

It is convenient for the following discussion to define

$$Re^{i\theta} \equiv \frac{ieI_{peak}Z_\perp(\omega_c)}{m_o\gamma c4\pi\nu\omega_s} \text{ and } \Delta^2 \equiv p_o^2\sigma_o^2. \tag{31}$$

With these changes we may rewrite Eq. (29)

$$1 = Re^{i\theta} \frac{i}{(e^{2\pi i\lambda}-1)} \int_0^{2\pi} e^{i\lambda t} e^{\Delta^2(\cos t-1)}. \tag{32}$$

At large growth rates ($\text{Im}\lambda > \Delta$), we can take the leading term to obtain

$$\lambda \simeq Re^{i\theta} \text{ or } \omega+\omega_y \simeq \frac{ieI_{peak}Z_\perp(\omega_c)}{m_o\gamma c4\pi\nu} \text{ for } \text{Im}(\omega/\omega_s) > \Delta. \tag{33}$$

It is straight forward to show using Eq. (32) that the growth rate is bounded;

$$\text{for } r < 1, \quad \text{Im}(\omega/\omega_s) \leq 1/\pi \ln\left(\frac{1}{1-r}\right), \tag{34}$$

$$\text{where } r \equiv Re^{-\Delta^2}\pi I_o(\Delta^2) \simeq R\sqrt{2\pi}/2\Delta \text{ for } \Delta >> 1 \tag{35}$$

$$\text{and } \text{Im}(\omega/\omega_s) \leq R. \tag{36}$$

For proton machines Δ is typically large. In Fig. 1 we plot the 2 upperbounds evaluated for $\Delta = 35$ which corresponds to the value for ISABELLE. As you see, there is a sharp rise at the intersection of the 2 bounds.

In order to obtain more precise results we must compute B. The most useful approach at this point is to plot a "stability diagram." From Equation (32) we have

$$b(\Delta,\lambda) = re^{i\theta} \text{ where } \frac{1}{b(\Delta,\lambda)} = 2\Delta B(\Delta^2,\lambda)/\sqrt{2\pi} \, . \qquad (37)$$

If we now plot a map of b in the complex plane identifying contours of constant growth rate, given the current and the impedance of a machine, we can find the point in the map which solves Eq. (37). Rewriting in terms of conventional quantities we have

$$b(\Delta,\lambda) = \frac{i\; e\; I_{peak} Z_\perp(\omega_c)}{4\nu\sqrt{2\pi}\; m_o\gamma c\; |\omega_s\sigma(\omega_c+\omega_\xi)|} \, . \qquad (38)$$

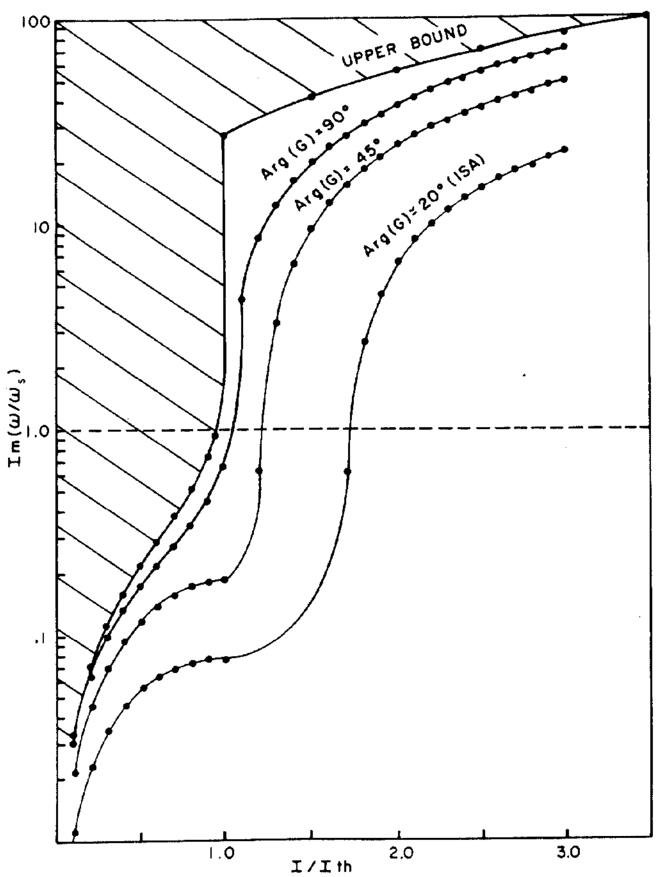

Fig. 1. Growth rate upperbound and growth rate vs current.

In Fig. 2 we plot contours of constant $Im(\lambda)$ for b $(35,\lambda)$. With the aid of this map we can plot the growth rate vs. current for any impedance. To illustrate this, consider an impedance which has equal real and imaginary parts. If we draw a ray in Fig. 2 at 45°, then the points of intersection yield a graph of β vs. r, or β vs I/Ith, Ith being the threshold current corresponding to r = 1 and $\beta = Im(\omega/\omega_s)$.

The curves given in Fig. 1 illustrate the above procedure. They have been extended below $Im(\lambda) = 1$ in order to suggest the slow blow-up regime; however, in order to calculate the transition to the slow blow-up regime correctly, one must take the negative frequencies into account also.

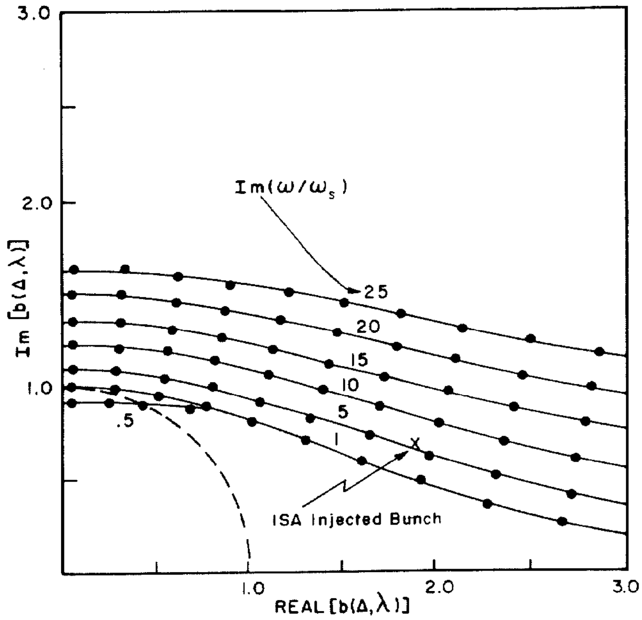

Fig. 2. Stability Diagram: Map of $b(\Delta = 35,\lambda)$.

Discussion

The "threshold" for the instability which we have here is given by

$$1 = \frac{e\; I_{peak}|Z_\perp(\omega_c)|}{4\sqrt{2\pi}\;\nu\;\eta\;\sigma_\varepsilon|\omega_c+\omega_\xi|m_o\gamma c} \quad \text{where } \eta\sigma_\varepsilon = \eta\left(\frac{\Delta p}{p}\right)_{rms} = \omega_s\sigma . \quad (39)$$

This is the threshold given by the unit circle in Fig. 2. As you see, the actual threshold for a given impedance will be at somewhat larger current due to the shape of the map of b in Fig. 2.

Notice the "phase transition" in Fig. 1 from the slow blow-up regime to the fast blow-up regime. This phase transition becomes the coasting beam threshold as $\omega_s \to 0$; however, for ω_s finite there will be a region of slow blow-up due to the head-tail effect[1] which we have suggested by extending the curve to $Im(\lambda) < 1$. (A detailed analysis of this transition will be given by the first author in an upcoming doctoral dissertation.)

To conclude, we take the impedance at ISABELLE to be a broadband impedance plus space charge. In order to estimate the broadband impedance we let $Z_\parallel/n = 10\ \Omega$ and scale the transverse impedance according to

$$Z_\perp = \frac{2c}{b^2}\,\frac{Z_\parallel}{n\omega_o}\,\frac{1}{[1-(x_o/b)^2]^2} \, , \qquad (40)$$

where x_o = beam center and b = chamber radius.

If we apply the above results, we find that for the injected bunch on the injection orbit I/Ith = 2. If we follow the curve in Fig. 1 corresponding to the ISA impedance, this leads to a growth rate ≈ 4msec (ω_s = 40/sec). This result is, however, sensitive to the location of the beam center. If the beam is centered, the growth rate is in the slow blow-up regime.

Acknowledgement

We would like to thank E.D. Courant and C. Pellegrini for many stimulating discussions.

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