© 1981 IEEE. Personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution to servers or lists, or to reuse any copyrighted component of this work in other works must be obtained from the IEEE.

IEEE Transactions on Nuclear Science, Vol. NS-28, No. 3, June 1981

APPROXIMATE PHYSICAL TREATMENT OF THE BEAM-BEAM EFFECTS

Lee C. Teng
Fermi National Accelerator Laboratory*
P.O. Box 500
Batavia, Illinois 60510

Abstract

When the beam-beam force is approximated as a periodic δ-function non-linear potential kicks, it is expected to excite a continuum(density of rational numbers) of resonances. The totality of resonance within a narrow "width" of the betatron oscillation tune contribute to a diffusion-like amplitude growth as if the kicks are random. A semi-quantitative formulation is derived and applied to electron colliding beams.

Introduction

Non-linear dynamics is fascinating because a purely deterministic system can exhibit apparently stochastic behaviors, thus leading to topics[1] such as singular mapping curves, Arnol'd diffusion, KAM theorem, overlapping resonances etc. But, so far, the relevance of these inherent features of non-linear dynamics to the observed behaviors of colliding beams has not been established. This prompted some investigators to suggest other approaches[2,3] to an explanation of the beam-beam effects. Moreover, on the conceptual level there are two rather strong arguments against the straightforward application of non-linear dynamics.

1. The observed beam-beam effect is an intrinsically statistical phenomenon. The exact temporal evolution of the non-linear motion of a particle, stochastic as it may be, is nevertheless time-reversible, whereas the beam-beam behavior exhibits irreversible diffusion-like characteristics. One should at the very outset introduce statistical averages into the equations of motion - a process similar to the transition from the Poincaré equation to the Liouville equation.

2. Real physical systems have noises. Although the effect of noise cannot be the whole picture it must form an integral part of the formulation.

In the following we sketch an attempt to formulate the beam-beam interaction in a statistical manner with noise as an integral part.

The Physical Picture

We consider the simplest case of a weak positron beam colliding head-on with a strong bunched electron beam. Reduced to bare essentials we have a single positron performing a stable linear one-dimensional transverse oscillation perturbed by a series of equally spaced extremely non-linear δ-function kicks.

The increment of the action invariant $W = \gamma x^2+2\alpha xx'+\beta x'^2$ due to a kick $\Delta x'$ is

$$\Delta W = \beta(\Delta x')^2 + 2(\alpha x+\beta x')\Delta x'. \qquad (1)$$

The kick can be expressed in terms of the "tune shift" ξ through

$$\xi = \frac{\beta}{4\pi}\frac{\Delta x'}{x} \quad \text{or} \quad \Delta x' = 4\pi\xi\frac{x}{\beta}.$$

If the successive kicks are random the second term in ΔW averages to zero and we get

$$\langle\Delta W\rangle = \beta\langle(\Delta x')^2\rangle = (4\pi\xi)^2\frac{\langle x^2\rangle}{\beta} = (4\pi\xi)^2\frac{W}{2} \qquad (2)$$

or

$$\left\langle\frac{\Delta W}{W}\right\rangle = \frac{1}{2}(4\pi\xi)^2 = 0.0079$$

where the last number corresponds to $\xi = 0.01$, an easily attainable value on all electron colliders. This is a very large number indeed, giving an e-fold increase in W in only $\frac{1}{0.0079} = 127$ kicks. The only reason that the positron motion can be stable is because these strong kicks are not random but periodic, and all evils are concentrated into resonances. On-resonance the effects of the kicks add coherently and the oscillation amplitude grows proportionally to the number of kicks. With random kicks the amplitude still grows as the square-root of the number of kicks. Off-resonance the effects of the kicks cancel systematically to give zero amplitude growth. The off-resonance cancellation is essential for the survival of the beam and is very exacting, especially for the very high order resonances. Any irregularity will upset the delicate cancellation. These time-domain descriptions are illustrated in Fig. 1.

For colliding beams both the non-linearity and the harmonics of the kicks extend to extremely high orders. The tune-space is covered dense by resonances(density of rational numbers), and the oscillation tune sits in a continuum of high order resonances even when all strong low order resonances are avoided. In fact, since the electron bunches are not exactly identical from collision to collision the kicks are not exactly periodic and all resonances have some "spreads" or "widths". This situation is equivalently described by assigning a natural "width" to the tune. This description avoids the possibility of confusing the "spread" of a resonance due to inexact periodicity of the kicks with the usual resonance width in non-linear dynamics. This description further suggests that the (small) portion of the kick-spectrum which is flat and equal in height to the part lying within the "width" of the tune will constitute a random series of (small) kicks in the time domain and cause the amplitude to grow. This is because a "white" spectrum in the frequency domain corresponds to a series of random signals in the time domain. The "natural width" is rather small, but the ever present hardware noise will contribute to the resonance spread and make the "total width" substantial. As described, the ultimate effect of noises, natural (beam) or external (hardware), is to take "strength" off from the resonance peaks and smear it in between resonances to form the "white" spectrum of a set of random kicks.

Partial Quantitative Formulation

The equation for for the y-motion of the positron is

$$y'' + k(s)y = -\frac{\partial V}{\partial y}\,\delta(s) \qquad (3)$$

where the force term on the right-hand-side is

*Operated by the Universities Research Association Inc., under contract with the U.S. Department of Energy.

0018-9499/81/0600-2402$00.75©1981 IEEE

expressible in closed form for round bi-Gaussian electron beam bunches and is given by[4]

$$-\delta(s)\,\frac{r_o N}{\gamma_p}\,\frac{d}{dy}\int_o^\infty dt\,\frac{1-e^{-\frac{y^2}{2(t+\sigma^2)}}}{t+\sigma^2}$$

$$= -\frac{r_o N}{\gamma_p \sigma^2}\left(\frac{1-e^{-\frac{y^2}{2\sigma^2}}}{\frac{y^2}{2\sigma^2}}\right) y\,\delta(s) \qquad (4)$$

where

σ = Gaussian standard deviation

$\gamma_p = \frac{E}{mc^2}$ of positron

$r_o = \frac{e^2}{mc^2}$ = classical radius of positron

N = total number of electrons.

The real electron beams are, however, not round but flat ribbons with $\sigma_x \gg \sigma_y$ (x = horizontal, y = vertical). Hence the vertical kicks are larger and dictate the intensity limit. Introducing the vertical "tune shift"

$$\xi_y = \frac{1}{4\pi}\,\frac{r_o N \beta_y}{\gamma_p \sigma_y(\sigma_x+\sigma_y)} \cong \frac{1}{4\pi}\,\frac{r_o N \beta_y}{\gamma_p \sigma_y \sigma_x}$$

and integrating Eq. (3) once to get the kick $\Delta y'$ we obtain

$$\Delta y' = -4\pi\xi_y\,\frac{\sigma_x}{\beta_y}\left(\frac{1-e^{-\frac{y^2}{2\sigma_y^2}}}{\frac{y^2}{2\sigma_y^2}}\,\frac{y}{\sigma_y}\right). \qquad (5)$$

The random part of these kicks in the sense discussed earlier will contribute to the growth of W_y. From Eqs. (2) and (5) we obtain

$$\langle\Delta W_y\rangle = \beta_y \langle(\Delta y')^2\rangle = \xi_y^2\,\frac{\sigma_x^2}{\beta_y}\,F\left(\frac{y^2}{2\sigma_y^2}\right) \qquad (6)$$

where

$$F(u) \equiv 32\pi^2\left[\frac{(1-e^{-u})^2}{u}\right]_{\text{random part}}. \qquad (7)$$

We drop some of the subscripts y as being understood and write

$$\left(\frac{dW}{dt}\right)_{bb} = f\xi^2 F\,\frac{\sigma_x^2}{\beta_y} \qquad (8)$$

where f = frequency of kicks. The total motion of the positron is, then, given by

$$\frac{dW}{dt} = Q - \frac{W}{\tau} + f\xi^2 F\,\frac{\sigma_x^2}{\beta_y} \qquad (9)$$

where

Q = growth due to quantum fluctuation

τ = synchrotron radiation damping time.

The function F has a maximum at u = 1.26 or y = 1.59 σ_y. The maximum "tune shift" ξ_{max} that can be obtained is given by the condition $\frac{dW}{dt} = 0$ at $F = F_{max}$, namely

$$f\xi_{max}^2\,F_{max}\,\frac{\sigma_x^2}{\beta_y} = \frac{W}{\tau} - Q. \qquad (10)$$

The energy dependence of the quantities are

$W \propto y^2 \propto \sigma_y^2 \propto E^2$;

$\frac{1}{\tau} \propto E^3$, hence $\frac{W}{\tau} \propto E^5$,

$Q \propto E^5$, coupled over from horizontal;

$\sigma_x \propto E^0$, because σ_x is aperture limited.

This gives

$$\xi_{max} \propto E^{5/2}. \qquad (11)$$

The energy dependence of the maximum luminosity L_{max} is related to that of ξ_{max} by[5]

$$L_{max} \propto E^2\,\xi_{max}^2 \propto E^7. \qquad (12)$$

Figs. 2 and 3 show the fits to the measured data from SPEAR[6] with

$$\xi_{max} = 0.01\,E^{5/2} \quad\text{and}\quad L_{max} = 0.03\,E^7$$

(L_{max} in $cm^{-2}sec^{-1}$ and E in GeV).

Discussions

1. Although the meaning was clearly stated no mathematical procedure has been developed to extract the "random part" in the definition, Eq. (7), of F. This involves contributions from both high order resonances and their "spreads" due to inexact periodicity or randomness of the kicks. It is possible that the mechanism proposed in Ref. 3 is appropriate. For the energy-dependence fit, however, all one needs is that $\left(\frac{dW}{dt}\right)_{bb}$ be proportional to ξ^2 and the proportionality factor be energy independent.

2. No explanation was given to the ultimate limit of $\xi_{max} \cong 0.05$. This limit is not statistical in nature and could well be given by the single particle non-linear dynamics. The conventional stochasticity limit due to overlapping of resonances is entirely consistent with the physical picture presented here.

References

1. See e.g. J. Moser, "Stable and Unstable Motion in Dynamical Systems", AIP Conf. Proc. No. 57, p. 222 (1979)
2. S. Kheifets, "Recent Experimental Results in the Beam-Beam Effects in Storage Rings and an Attempt of Their Interpretation", Proc. of Beam-Beam Int. Seminar, SLAC-PUB-2624, CONF-8005102, p. 40 (1980)
3. D. Neuffer, A. Riddiford and A. Ruggiero,"Diffusion Enhancement by the Beam-Beam Interaction", Paper L-26 (F-36) this conference

4. B.W. Montague, "Calculation of Luminosity and Beam-Beam Detuning in Coasting Beam Interaction Regions", CERN/ISR-GS/75-36 (1975)
5. L.C. Teng, "Beam-Beam Phenomenology", Proc. of Beam-Beam Int. Seminar, SLAC-PUB-2624, CONF-8005102, p. 99 (1980)
6. H. Wiedemann, "Beam-Beam Effect and Luminosity in SPEAR", Proc. of Beam-Beam Int. Seminar, SLAC-PUB-2624, CONF-8005102, p. 33 (1980)

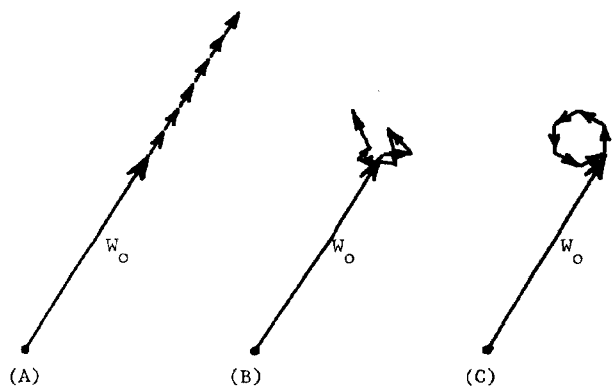

Fig. 1. Effects of kicks in the time domain (A) on-resonance (B) random (C) off-resonance

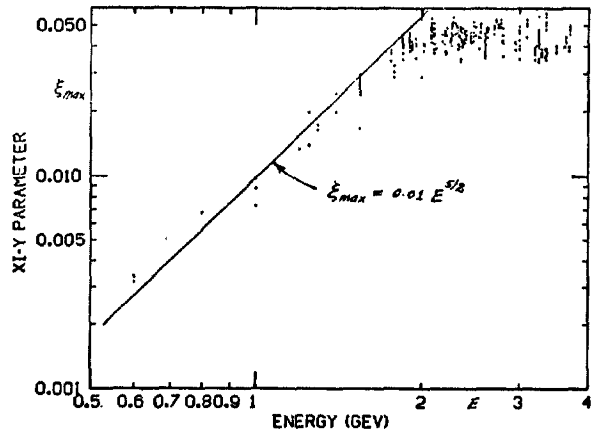

Fig. 2. Maximum vertical tune shift versus energy in SPEAR.

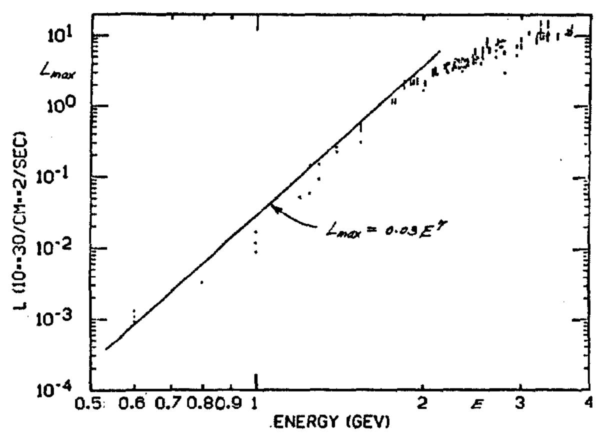

Fig. 3. Maximum luminosity versus energy in SPEAR.