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REGULARIZATION IN FOURIER-SYNTHESIS 2-D IMAGE RECONSTRUCTION*

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ABSTRACT

A filtering procedure for computed tomography is formulated. It is derived from the general regularization theory and contains an adjustable regularization parameter. The latter can be found self-consistently and corresponds to a given noise level. The method is illustrated by applying it to the reconstruction of phantoms in computer experiments.

Improvement of resolution in image reconstruction from projections requires development of refined filtering procedures. It is our purpose to present here such a filtering procedure based on the regularization method[1-4]. The method is specially designed to handle ill-posed problems, i.e. those that tend to instability when noisy data are used. When applied to a deconvolution problem the said method is an approximation to Wiener's optimal linear filtration.

The common formulation of the Fourier-synthesis method in 2-D image reconstruction is as follows[5]:

$$f(r,\phi) = \int_0^{\pi} \int_{-\infty}^{\infty} |k| G(k,\theta) \exp[2\pi i k r \cos(\phi-\theta)]\, dk\, d\theta \qquad (1)$$

Here $f(r,\phi)$ is the object density function in polar coordinates and $G(k,\theta)$ the one-dimensional Fourier transform of the projection $g(\ell,\theta)$ of the object onto a line which forms an angle θ with the polar axis.

The regularization approach is commonly applied to integral equations of the first kind. It is particularly effective for convolution-type equations where the Fourier transformation can be used. In the case of computed tomography a convolution-type equation appears in the filtered summation-image method and it can be presented in the Fourier plane as follows:

$$F(k_x,k_y) = k \cdot S(k_x,k_y) \qquad (2)$$

Here $F(k_x,k_y)$ and $S(k_x,k_y)$ are the Fourier transforms of the object density function and of the summation image; $k = (k_x^2 + k_y^2)^{1/2}$. For details, see for example, Ref. (6). Barrett et al.[6,7] proved that the one-dimensional filter function in the Fourier-synthesis method is equal to the two-dimensional filter in the filtered summation-image method provided that the polar radius in the Fourier plane is treated as a one-dimensional variable. The two-dimensional filter is assumed to be circularly symmetric. Eqs. (1) and (2) illustrate this theorem. Any additional filtering applied to improve signal-to-noise ratio is governed by the same theorem.

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A regularized solution of a two-dimensional convolution-type equation was obtained by Goncharskii et al.[8] The Fourier transform of their regularized solution is:

$$F_\alpha(k_x,k_y) = k \cdot S_\delta(k_x,k_y)/[1 + \alpha k^2(1+ak^4)] \qquad (3)$$

Here the notation $S_\delta(k_x,k_y)$ is used instead of $S(k_x,k_y)$ to underline the fact that it is a Fourier transform of an error-containing function so that

$$||s_\delta(x,y) - s_{true}(x,y)|| \le \delta \qquad (4)$$

The coefficient a is related to a Sobolev norm chosen in the object space. It is taken here to be equal to unity. The object diameter is also assumed to be unity.

The inverse Fourier transform of Eq. (3) yields the approximate solution $f_\alpha(x,y)$. According to Goncharskii et al.[8] it converges uniformly to the exact solution when $\delta \to 0$. In Eq. (3) α is a positive parameter inherent in the regularization method and is called the regularization parameter. It defines the degree of smoothing in the approximate solution and is a function of the error δ. It can be proved[9] that α is a continuous and monotonically growing function of δ. ($\alpha \to 0$ when $\delta \to 0$).

On the basis of the above we can proceed with our main objective i.e., to introduce the regularization procedure into the Fourier-synthesis method. According to Barrett's theorem, the filter function in Eq. (3) can be substituted into Eq. (1) as follows:

$$f_\alpha(r,\phi) = \int_0^{\pi} \int_{-\infty}^{\infty} |k| G_\delta(k,\theta)/[1+\alpha k^2(1+k^4)] \times$$

$$\times \exp[2\pi i k r \cos(\phi-\theta)]\, dk\, d\theta \qquad (5)$$

In this last case the error for the data should be defined as:

$$\int_0^{\pi} \int_{-\infty}^{\infty} [g_\delta(\ell,\theta) - g_{true}(\ell,\theta)]^2 d\ell\, d\theta \le \delta^2 \qquad (6)$$

In model calculations δ can be evaluated directly from Eq. (6). However, in the case of real data some statistical estimation of δ is necessary. The use of difference tables of the data for the estimation of the noise level may be useful[10]. Alternatively the range of the samples can be used.

We evaluate the regularization parameter α by the so-called residual error method[9]. By this method α is chosen to satisfy the following equation:

$$\int_0^{\pi} \int_{-\infty}^{\infty} [g_\alpha(\ell,\theta) - g_\delta(\ell,\theta)]^2 d\ell\, d\theta = \delta^2 \qquad (7)$$

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Here $g_\alpha(\ell,\theta)$ is the "reconstructed experimental data" as computed from $f_\alpha(r,\phi)$. It follows from Eq. (7) that the regularization parameter must be chosen to match the precision of the data. It represents a certain compromise between over-smoothing (big α's) and noisy solutions (small α's).

We can now transform Eq. (7) into the Fourier domain in order to enhance effectiveness of the computation of α. Using Plancherel's theorem we have:

$$\int_0^\pi \int_{-\infty}^{\infty} [G_\alpha(k,\theta) - G_\delta(k,\theta)]^2 dk d\theta = \delta^2 \qquad (8)$$

From Eqs. (1) and (5) it may be seen that:

$$G_\alpha(k,\theta) = G_\delta(k,\theta)/[1 + \alpha k^2(1+k^4)] \qquad (9)$$

The substitution of Eq. (9) into (8) finally yields

$$\int_0^\pi \int_{-\infty}^{\infty} \alpha^2 k^4 (1+k^4)^2 |G_\delta(k,\theta)|^2 / [1 + \alpha k^2(1+k^4)]^2 dk d\theta = \delta^2 \qquad (10)$$

This is our final equation for the regularization parameter α .

Eqs. (5) and (10) define a noise-matched filtering procedure for a 2-D image reconstruction from projections in the framework of the Fourier-synthesis method. The procedure is based on the application of the regularization method to the computation of the filtered summation-image as a two-dimensional deconvolution problem. The known connection between the filtered summation-image and the Fourier-synthesis methods enabled us to apply the regularization procedure to the Fourier-synthesis method.

Computer Model Calculations

Four examples of computer-simulation of the reconstruction of mathematical phantoms by the regularized Fourier-synthesis algorithm are presented in Figs. (1a-d). The phantoms, as functions of two variables, were assumed to possess rotational symmetry. Their central cross-sections are represented by full lines in the figures. The reconstructed phantoms are given by the broken lines. The projections $g(\ell,\theta)$ were calculated from the mathematical phantoms. A gaussian noise with a standard deviation proportional, at each point, to the value of $g(\ell,\theta)$ and with a zero mean was added to the projections. The noise level of the data can thus be characterized by a single number: the percentage error. The reconstructionwas carried out on a 2048×2048 lattice. The projections were sampled in 2048 points along the diameter for the variable ℓ . (18 equally spaced views were taken for θ .) This choice was dictated by the fact that the computations were performed on the basis of Eq. (5) where a Fourier transform had to be performed on a function with a discontinuity in its first derivative. The value of δ^2 was calculated by means of Eq. (6) and was taken as three times the integral on the left hand side of Eq. (6). This factor was chosen somewhat arbitrarily; smaller factors resulted in too small α's and led to noisy solutions, while factors bigger by 1 or 2 did not influence substantially the reconstructed picture. The regularization parameter α was calculated from Eq. (10) by successive approximations as follows: an initial value of α was chosen and substituted into the left-hand side of Eq. (10). If for example, the computed expression exceeded the given δ^2 it meant that the chosen value of α corresponded to a bigger value of the right-hand side of Eq. (10) and vice versa. The next trial was then performed with a smaller α since as mentioned before, α is a monotonically growing function of δ. This procedure satisfactorily converges to a single solution provided that δ is not too big.

It may be noted from Figs. (1a-d) that the reconstruction is reasonably good for a 0.1% noise level. The fine details of the picture are reconstructed and there is practically no high frequency noise present. With a 0.5% noise level (Fig. 1b) almost all fine details are lost and only the general features of the picture are preserved. Nevertheless the reconstructed picture remains practically noiseless.

If a good estimation of the regularization parameter α is available from previous similar computations or from appropriate computer experiments, it is not necessary to solve Eq. (10). Model calculations indicate (see Fig. 1a,c,d) that the regularization parameter α is mostly dependent on the noise level and is to some extent insensitive to changes in the structure of the objects to be reconstructed.

In general it can be stated that the regularized Fourier-synthesis method provides a refined filtration operation for reconstruction problem from noisy data. The filtration is based on the use of a parameter that must be adjusted to the noise-level of the data. This adjustment can be performed automatically so that the whole method is self-contained.

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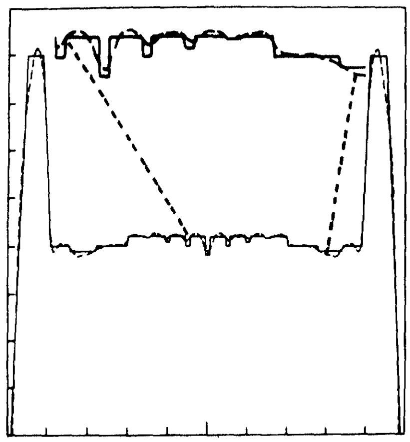
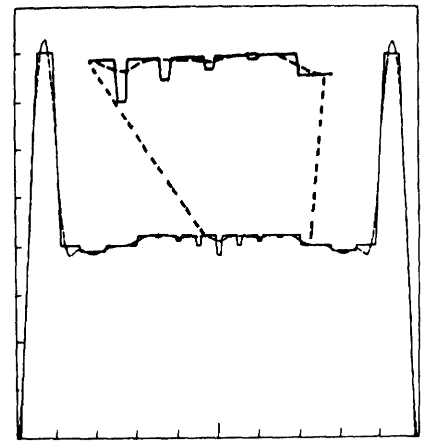
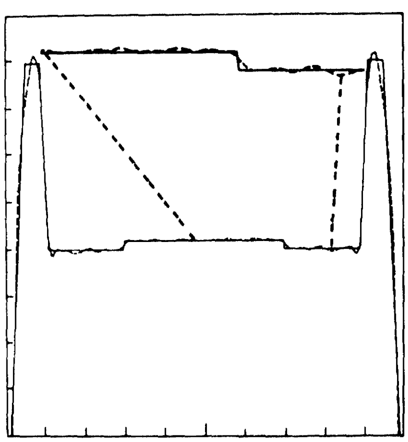


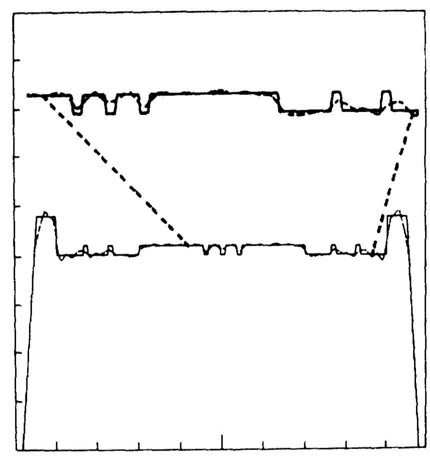
a) Noise level 0.1%.
Regularization parameter 6.69×10^{-10}



b) Noise level 0.5%.
Regularization parameter 7.08×10^{-8}



c) Noise level 0.1%.
Regularization parameter 6.98×10^{-10}



d) Noise level 0.1%.
Regularization parameter 6.98×10^{-10}

Fig. 1. Reconstructed phantoms are given by broken lines. The central cross-sections of the original phantom are given by full lines. The insertions represent enlarged sections of the phantoms.