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THE TRANSIENT RESPONSE OF A MICROWAVE CAVITY

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Abstract

The classical analysis of the transient response of a microwave cavity has experienced a substantial re-examination in recent years for the reason that, owing to improved technology, it is possible to observe some aspects of the transient response formerly obscured and partly because of the misguided desire of reconciling lumped circuit analysis with the distributed circuit (microwave) solution.

In this paper the transient response is considered; secondly, the field solution is determined from the differential equation of the equivalent circuit. An apparent discrepancy in the solutions is resolved. The analysis also provides a quantitative explanation of the reflected power pattern during the RF pulse.

Preliminary Remarks

There are some global remarks which may be made before a detailed examination of the system being analyzed.

The power that enters a cavity through the coupling mechanism (P_0) will (in the case where there is no beam loading) be partitioned into joulean losses (P_L) and stored energy (U).

$$P_o = P_L + \frac{dU}{dt} \qquad (1)$$

By definition of the Q of the cavity (in this case "loaded" because of the coupling mechanism) as the ratio of stored energy to energy dissipated in one radian,

$$Q_L = \frac{\omega U}{P_L} \qquad (2)$$

the above becomes

$$\frac{dU}{dt} + \frac{\omega U}{Q_L} - P_o = 0 \qquad (3)$$

This expression may be integrated by separation of variables; with boundary conditions U = 0, t = 0,

$$U(t) = \frac{P_o Q_L}{\omega}\left(1 - e^{-\omega t/Q_L}\right) \qquad (4)$$

Physically, the ultimate stored energy will be $U = P_0 Q_L/\omega$, ie, the Q of the circuit is given by $Q = \omega U/P_0$, as defined earlier (where ultimately all the input power goes into sustaining losses).

The power that enters the cavity depends upon the mismatch at the coupling mechanism,

$$P_o = (1 - |\rho|^2)\, P_i \qquad (5)$$

where ρ is the voltage reflection coefficient and P_i the incident power. In terms of the more convenient measure of mismatch, VSWR (σ),

$$|\rho| = \frac{\sigma - 1}{\sigma + 1} \qquad (6)$$

This VSWR is also the coupling coefficient at resonance, β, (or its reciprocal in the undercoupled case). Hence the input power to the cavity is

$$P_o = \frac{4\beta}{(1+\beta)^2} P_i \qquad (7)$$

Incidentally, it will be noted that the maximum power transfer occurs at critical coupling ($\beta = 1$), but this is not the condition for maximum cavity voltage.

The cavity voltage in the steady-state at resonance,

$$V_c = \sqrt{2 R_L P_o} = \sqrt{\frac{8\beta P_i R_o}{(1+\beta)^3}} \qquad (8)$$

which has a maximum when $\beta = 0.50$.

The reader may object to the apparently simplistic analysis in deriving Eq (4), in that the power flow is not a constant, being a product of a sinusoidal voltage and current (essentially a Poynting vector) at the coupling mechanism,

$$P(t) = V_m I_m \cos^2 \omega t \qquad (9)$$

On the other hand, the transient response arises from the homogeneous equation (no driving force) and could have been gotten from

$$\frac{dU}{dt} + \frac{\omega U}{Q_L} = 0$$

with appropriate boundary conditions and interpretive insight. The solution is, in either case, physically the envelope of the oscillatory response, as the disposition of the energy only is being considered and this energy oscillates between completely electric and magnetic form, but the envelope represents the total quantity. The gap voltage amplitude (squared) has a linear relation to the stored energy,

$$V_c^2 = 2\frac{R}{Q}\omega U = 2R_L P_o\left(1 - e^{-\omega t/Q_L}\right) \qquad (10)$$

Circuit Analysis

In the phenomenological description of microwave circuits it is demonstrated that a transcription of Maxwell's equations is possible into matrix algebraic equations ; this transcription is quite rigorous, and is a result of the fact that Maxwell's equations belong to the Sturm-Liouville class . Thus, any approximations that appear in (lumped) circuit theory arguments are a consequence not of the transcription but as a result of ignoring some rows or columns of the impedance matrix provided by field theory.

The above remarks mean that the response of a cavity to excitation may be represented by matrix algebra methods (that is, in terms of lumped circuit elements); it does not mean that specific parts of a cavity may be identified with conventional circuit elements, and therefore, that the uniquely definable voltages and currents in the lumped circuit are not specified in the distributed circuit.

If the input impedance of a cavity is plotted in the plane of its terminals (plane of the detuned short) the frequency trajectory of impedance points (on, say, a Smith chart) will be represented as shown on Fig. 1 (a); if the data is plotted one-quarter wavelength from its terminals (in the plane the detuned open) the trajectory is also shown in Fig. 1 (b). By the equivalent of Foster's Reactance Theorem for circuits containing resistance, clearly it is not possible to decide if the circuit is a series or parallel resonant circuit; this is, of course, a consequence of the transforming property of a transmission line, but as a result it is not necessary to decide what is the "true" nature of the circuit.

Solution of the Equivalent Circuit Transient Case

With the lumped circuit equivalent of the resonant cavity shown in the figure

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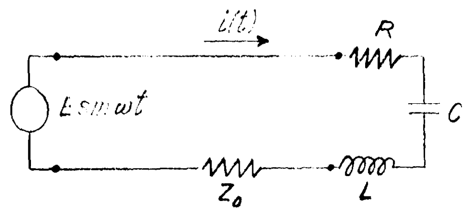

the equation of dynamic equilibrium is

$$L\frac{di}{dt}+i(R+Z_0)+\frac{1}{C}\int i\,dt=Ee^{j\omega t} \quad (11)$$

for the case of forced oscillation. The solution of this equation becomes somewhat relevant in the resolution of the transient case so it will be given here. Eq (11) may be differentiated

$$\frac{d^2i}{dt^2}+(R+Z_0)\frac{di}{dt}+\frac{i}{LC}=\frac{\omega E}{L}\cos\omega t \quad (12)$$

Try the solution

$$\begin{aligned} i &= I\cos(\omega t+\phi)\\ di/dt &= -\omega I\sin(\omega t+\varphi)\\ d^2i/dt^2 &= -\omega^2 I\cos(\omega t+\varphi) \end{aligned} \quad (13)$$

Then,

$$\left(-\frac{\omega E}{L}-I\left(\left(\omega^2-\frac{1}{LC}\right)\cos\varphi+\frac{\omega(R+Z_0)}{L}\sin\varphi\right)\cos\omega t+I\left(\left(\omega^2-\frac{1}{LC}\right)\sin\varphi-\frac{\omega(R+Z_0)}{L}\cos\varphi\right)\sin\omega t=0 \quad (14)$$

which will be satisfied for all values of t when the coefficients vanish,

$$\tan\varphi=\frac{(R+Z_0)}{\left(\frac{1}{\omega C}-\omega L\right)} \quad (15)$$

$$I=\frac{E}{\left(\frac{1}{\omega C}-\omega L\right)\cos\varphi+(R+Z_0)\sin\varphi}=\frac{E}{\sqrt{\left(\frac{1}{\omega C}-\omega L\right)^2+(R+Z_0)^2}}$$

The damping obviously plays no part in determining the driving frequency of maximum response of the circuit, which clearly occurs when $\omega = 1/LC$, and is the intended driving frequency.

The transient part of the solution is the solution of the homogeneous or force-free equilibrium condition,

$$L\frac{di}{dt}+i(R+Z_0)+\frac{1}{C}\int i\,dt=0 \quad (16)$$

We assume a solution of the form

$$i=Ae^{pt} \quad \left(\frac{di}{dt}=ip,\ \int i\,dt=\frac{i}{p}\right) \quad (17)$$

Substituting into Eq (12),

$$\left(Lp+(R+Z_0)+\frac{1}{Cp}\right)i=0 \quad (18)$$

which, for a non-trivial solution (ie, $i\neq 0$) leads to the further condition

$$Lp+(R+Z_0)+\frac{1}{Cp}=0 \quad (19)$$

which is the determinantal equation of the system; the roots of this equation are the natural modes of the system. Solving this quadratic equation we find two roots:

$$p_{1,2}=-\frac{R+Z_0}{2L}\pm\sqrt{\left(\frac{R+Z_0}{2L}\right)^2-\frac{1}{LC}} \quad (20)$$

As we are only interested in highly oscillatory cases,

$$\frac{1}{LC}>>\left(\frac{R+Z_0}{2R}\right)^2 \quad (21)$$

and the roots may be written

$$p_{1,2}=-\frac{R+Z_0}{2R_L}\pm j\sqrt{\frac{1}{LC}-\left(\frac{R+Z_0}{2L}\right)^2} \quad (22)$$

The ratio $\frac{\omega L}{R+Z_0}$ is demonstrably the ratio of the stored energy to time-average losses per radian, hence we may put

$$\frac{Q_L}{\omega}=\frac{L}{R+Z_0} \quad (23)$$

so that

$$p_{1,2}=-\frac{\omega_0}{2Q_L}\pm j\sqrt{\omega_0^2-\left(\frac{\omega_0}{2Q_L}\right)^2} \quad (24)$$

where $\omega_0 = 1/LC$ is the frequency of the mode of oscillation in the absence of damping. The frequency of free oscillation in the damped case is given by

$$\omega_n=\omega_0\sqrt{1-\left(\frac{1}{2Q_L}\right)^2} \quad (25)$$

that is, the natural frequency of the circuit is less than it would have been in the absence of loss.

The complete solution of Eq. (11) includes that of Eq. (16), since the homogeneous case is complementary to the non-homogeneous; that is, the complete solution is the sum of the transient and steady-state responses; then we may evaluate the integration constant from the initial conditions. Many writers consider the solution to consist of a steady-state term and two transient terms, one an exponentially damped sinusoid, the other a damped cosinusoid; at $t = 0$ the transient is equal in magnitude and opposite in sign to the steady-state. Physically, there is only one natural mode, so

$$\begin{aligned} Ae^{pt} &= A_1e^{p_1t}+A_2e^{p_2t}\\ &= Ae^{-\omega_0t/2Q_L}\left(e^{j\omega_nt}-e^{-j\omega_nt}\right)\\ &= Ae^{-\omega_0t/2Q_L}\sin\omega_nt \end{aligned} \quad (26)$$

where at $t = 0$, $i = 0$ so $A_1 = -A_2 \equiv A/2$.

So, the complete solution is

$$\begin{aligned} i &= \frac{E}{R}\left(\sin\omega_0t-e^{-\omega_0t/2Q_L}\sin\omega_nt\right)\\ &= \frac{E}{R}\sqrt{1-2e^{-\omega_0t/2Q_L}\cos(\omega_n-\omega_0)t+e^{-\omega_0t/Q_L}}\ \sin\omega_0t\\ &\doteq \frac{E}{R}\left(1-e^{-\omega_0t/2Q_L}\right)\sin\omega_0t \end{aligned} \quad (27)$$

since in the highly oscillatory case $(\omega_n-\omega_0)$ vanishes, and from the multiplicity of possible solutions (resulting from initial conditions) we have taken that the applied voltage is passing through a node at $t = 0$ and is at the resonant driving frequency.

Note the voltage between two points in a microwave cavity is of the form $V^2 = 2R_{sh}P$ where R_{sh} is the shunt resistance. (The 2 arise from a convention among microwave engineers to express peak voltages, but power is always an effective average.) Similarly, the circuit current is of the form $P = I^2R_{se}$ where R_{se} is the series resistance. The term $\sqrt{R_{sh}R_{se}}$ does not differ from the impedance of free space by more than a geometrical constant for any mode; hence, the form of the cavity voltage does not differ from that of the circuit current.

It will, doubtless, not escape the reader's notice that something has gone wrong here; Eq. (10) does not agree with Eq. (27). The cause of the discrepancy lies in the fact that the power to the cavity is time dependent, that is, Eq. (1) is basically non-linear,

$$\frac{dU}{dt} + \frac{\omega U}{Q_L} = P_o(t) \qquad (28)$$

If the input power to the cavity (P_o) is taken as

$$P_o(t) = \frac{4\beta P_i}{(1+\beta)^2}\left(1 - e^{-\omega t/2Q_L}\right) \qquad (29)$$

it will be obvious that a singular solution of Eq.(28) is

$$U(t) = \frac{4\beta P_i}{(1+\beta)^2}\,\frac{Q_L}{\omega}\left(1 - e^{-\omega t/2Q_L}\right)^2 \qquad (30)$$

from which the cavity voltage is

$$V(t) = \sqrt{2R_L\,\frac{4\beta P_i}{(1+\beta)^2}}\left(1 - e^{-\omega t/2Q_L}\right) \qquad (31)$$

Physically, the explanation of Eq. (29) is that the input power to the cavity is only constant at steady-state (cf. Eq. 7). The applied field at the coupler does not "see" the steady-state equivalent circuit of the cavity in the first instants. That this is so may be seen from an oscillogram of reflected power when a unit step function of RF power is applied to a cavity, Fig. 2, an effect well-known to experimentalists (initially a resonant cavity is a short circuit).

The Discharge Transient

When the cavity circuit is in steady-state ($t \to \infty$) the stored energy (U_o)

$$U_o = \frac{P_o Q_L}{\omega} = \frac{4\beta P_i}{(1+\beta)^2}\,\frac{Q_L}{\omega} \qquad (32)$$

This follows from the definition of $Q = \omega U_o/P_o$, where all the input power ultimately supports losses.

If the cavity were without coupling the oscillatory discharge damping would attenuate the stored energy,

$$\frac{dU}{dt} = -\frac{\omega U}{Q_o} \qquad (33)$$

the solution of which is

$$U(t) = \frac{P_o Q_o}{\omega} e^{-\omega t/Q_o} \qquad (34)$$

but with coupling to the outside world, (the real case), the rate of energy loss is greater than Eq. (33) indicates. Then

$$\frac{dU}{dt} = -P_L - P_e$$

ie.,

$$\frac{dU}{dt} + \frac{\omega U}{Q_L} = -P_e \qquad (35)$$

where P_e is the power radiated through the coupling aperture. Transcending the general solution, realism requires

$$U = \frac{P_o Q_L}{\omega} e^{-2\omega t/Q_L} \qquad (36)$$

$$P_e = P_o e^{-2\omega t/Q_L} \qquad (37)$$

as may be verified by substitution.

It may appear that the form of the power flux in the solution of Eqs. (28) and (37) was gratuitously chosen to produce the desired result; such is not the case. But it is true insofar as the solutions satisfy the Lorenz-Lorentz Thermodynamic Theorem (Only half the stored energy is recoverable from a storage system as free energy). The stored energy at steady-state is $U_o = P_o Q_L/\omega$; the radiated energy is

$$\int_0^\infty P_e\,dt = \frac{P_o Q_L}{2\omega} \qquad (38)$$

Until P(t) is defined there is no algorithm for the solution of Eq. (28). The nature of P(t) must obviously be found in the physical process. Although an analysis of oscillograms of the transient process suggests itself as a basis for understanding the mathematical physics of Eq. (28), that procedure leaves us without a physical model of the process. The physics of the transient response of a finite length of lossless transmission line with a resonant cavity termination is similar to the classical antenna problem. This is a first order reaction (one in which the rate of reaction is directly, or inversely, proportional to the amount of what is changing), so that one expects P(t) to contain an exponential function; little more can be said.

Of interest, but not discussed above, is the transient nature of the reflection coefficient to the input pulse. Since $P_i = P_r + P_c$, where P_r is the reflected power, we may solve for P_r

$$P_r = P_i\left(1 - \frac{4\beta}{(1+\beta)^2}\left(1 - e^{-\omega t/2Q_L}\right)\right) \qquad (39)$$

Then, the reflection coefficient during the input pulse

$$\rho = \frac{V_r}{V_i} = \sqrt{\frac{P_r}{P_i}} = \sqrt{\frac{(1-\beta)^2 + 4\beta e^{-\omega t/2Q_L}}{(1+\beta)^2}} \qquad (40)$$

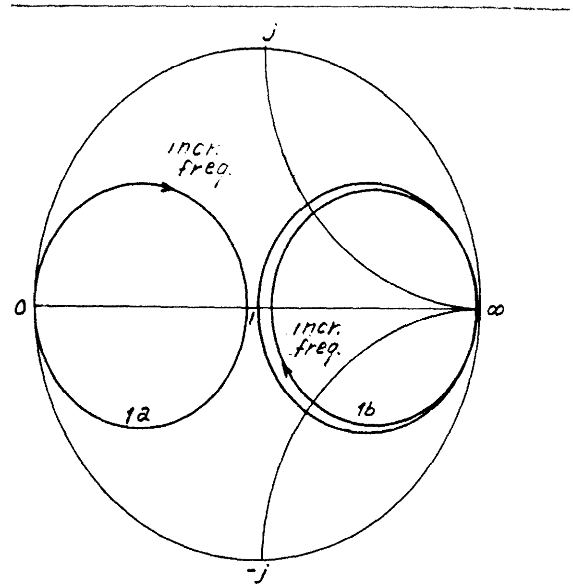

Fig. 1 Locus of impedance points as a function of frequency:
(a) parallel resonance,
(b) series resonance.

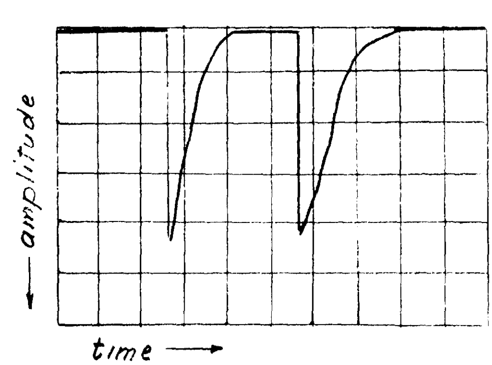

Fig. (2). Reflected power from a resonant cavity $f_o = 208$ mcs, $Q_o = 24{,}000$ $Q_L = 12{,}000$, $\beta \doteq 1.0$ (traced from a polaroid oscillogram)