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Introduction

Earlier calculations on the same structural composite assumed media possessing isotropic symmetry.¹ The present calculation utilizes elastic constants possessing locally orthotropic symmetry² (reflection symmetry about three orthogonal planes). In particular, this permits a more realistic representation of the subdivision of superconductor into cables. In keeping with the orthotropic symmetry, three thermal strain constants are used for each cylindrical region, one for each principal axis of symmetry. Again a pre-load condition is introduced using the notion of dislocation.³

Equilibrium

Generally the dipoles of interest are long compared to transverse dimensions and hence only two dimensional calculations are needed. The effect of magnet ends on the body then is handled in an integral manner using the virial theorem.⁴ In elasticity language this is designated as a generalized plane strain approximation. Cylindrical coordinates seem most appropriate for the problem at hand. Hence if \vec{u} is the elastic displacement vector the generalized plane strain approximation limits the functional dependence to

$$u_r(r, \theta) \quad u_\theta(r, \theta) \quad u_z = z \epsilon_{zz} \quad (\epsilon_{zz} = \text{constant}). \quad (1)$$

If $\vec{\sigma}$ designates the stress tensor and $\vec{J} \times \vec{B}$ the Lorentz force then the equilibrium conditions are given by³

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} - J_z B_\theta = 0, \quad (2)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r} \sigma_{r\theta} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + J_z B_r = 0, \quad (3)$$

under the functional limitations described in Eq. (1).

Relation of Stress to Strain

Since the ends are far removed from the region of interest the shear stresses σ_{rz} and $\sigma_{\theta z}$ may be neglected. Consequently two of the nine elastic constants representing orthotropic symmetry are not needed. Thus²

$$\sigma_{rr} = C_{11}(\epsilon_{rr} - k_1) + C_{12}(\epsilon_{\theta\theta} - k_2) + C_{13}(\epsilon_{zz} - k_3) \quad (4)$$

$$\sigma_{\theta\theta} = C_{12}(\epsilon_{rr} - k_1) + C_{22}(\epsilon_{\theta\theta} - k_2) + C_{23}(\epsilon_{zz} - k_3) \quad (5)$$

$$\sigma_{zz} = C_{13}(\epsilon_{rr} - k_1) + C_{23}(\epsilon_{\theta\theta} - k_2) + C_{33}(\epsilon_{zz} - k_3) \quad (6)$$

$$\sigma_{r\theta} = C_{44} \epsilon_{r\theta}, \quad (\text{standard notation uses } C_{66}) \quad (7)$$

where k_1, k_2, k_3 represent the thermal strains from room temperature where they are zero to some other operating temperature.

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Relation of Strain to Displacement

In cylindrical coordinates strain is defined³ as

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \quad (8)$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \quad (9)$$

$$\epsilon_{r\theta} = \frac{1}{2} \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \quad (10)$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} \quad (11)$$

Force on Thick Cosine Theta Conductor

As indicated previously¹ for $J_z = J_0 \cos \theta$ one has

$$J_z B_\theta = \frac{\pi}{3} J_0^2 [4r - 3\lambda - b^3 r^{-2}] (1 + \cos 2\theta), \quad (\text{emu}) \quad (12)$$

$$J_z B_r = \frac{\pi}{3} J_0 [2r - 3\lambda + b^3 r^{-2}] \sin 2\theta, \quad (\text{emu}) \quad (13)$$

where for convenience in these and subsequent formulas

$$\lambda = c + \frac{1}{3}(c^3 - b^3)r_s^{-2}, \quad (14)$$

the radius of the shield being r_s .

Form of Solution

First express stress in terms of strain using Eqs. (4-7) and then strain in terms of displacement using Eqs. (8-11). Then the equilibrium condition of Eqs. (2-3) after noting the Lorentz forces in Eqs. (12-13) indicates that the displacements have the form

$$u_r = P_0(r) + P_2(r) \cos 2\theta \quad u_\theta = Q_0(r) \theta + Q_2(r) \sin 2\theta. \quad (15)$$

Solution of Homogeneous Equations

Special solutions related to a dislocation³ ($Q_0 \neq 0$) are

$$P_0 = \frac{C_{12} - C_{22}}{C_{11} - C_{22}} Gr \quad Q_0 = Gr. \quad (16)$$

Regular solutions ($Q_0 = 0$) become

$$P_0 = Ar^p + Br^{-p} \quad Q_0 = 0, \quad (17)$$

where

$$p = \sqrt{\frac{C_{22}}{C_{11}}}. \quad (18)$$

And

$$P_2 = \gamma_1 Cr^{s_1} + \gamma_2 Dr^{s_2} + \gamma_3 Er^{s_3} + \gamma_4 Fr^{s_4} \quad (19)$$

$$Q_2 = Cr^{s_1} + Dr^{s_2} + Er^{s_3} + Fr^{s_4}, \quad (20)$$

where $(s_1 s_2 s_3 s_4)$ are the four roots of

$$s^4 - \left(8 \frac{C_{22}}{C_{44}} - 8 \frac{C_{12}^2}{C_{11} C_{44}} - 8 \frac{C_{12}}{C_{11}} + \frac{C_{22}}{C_{11}} + 1\right) s^2 + 9 \frac{C_{22}}{C_{11}} = 0 \quad (21)$$

and

$$\gamma_k = \frac{\frac{1}{2}(s_k^2 - 1)C_{44} - 4C_{22}}{(2C_{12} + C_{44})s_k + 2C_{22} + C_{44}}. \quad (22)$$

Particular Solution

The particular solution may be thought of as an incremental displacement to be added to the solution of the homogeneous equations. However, since ϵ_{zz} is an unknown this contribution of the particular solution will be transferred to the homogeneous category. After assembling all the contributions one has

$$\begin{aligned} \Delta u_r = & \frac{\pi}{3} J_0^2 \left[\frac{4r^3}{9C_{11} - C_{22}} - \frac{3\lambda r^2}{4C_{11} - C_{22}} + \frac{b^3}{C_{22}} \right] \\ & + \frac{(C_{11} - C_{12})k_1 + (C_{12} - C_{22})k_2 + (C_{13} - C_{23})k_3}{C_{11} - C_{22}} \\ & + \frac{\pi J_0^2}{3\Delta(3)} [12C_{12} - 20C_{22} + 20C_{44}] r^3 \cos 2\theta \\ & + \frac{\pi J_0^2}{3\Delta(2)} [-12C_{12} + 18C_{22} + \frac{15}{2}C_{44}] \lambda r^2 \cos 2\theta \\ & + \frac{\pi J_0^2}{3\Delta(0)} [2C_{22} - \frac{1}{2}C_{44}] b^3 \cos 2\theta, \quad (23) \end{aligned}$$

$$\begin{aligned} \Delta u_\theta = & \frac{\pi J_0^2}{3\Delta(3)} [-18C_{11} + 24C_{12} + 10C_{22} + 20C_{44}] r^3 \sin 2\theta \\ & + \frac{\pi J_0^2}{3\Delta(2)} [12C_{11} - 12C_{12} - 9C_{22} - 15C_{44}] \lambda r^2 \sin 2\theta \\ & + \frac{\pi J_0^2}{3\Delta(0)} [-C_{22} + C_{44}] b^3 \sin 2\theta, \quad (24) \end{aligned}$$

where

$$\Delta(p) = \frac{1}{2}C_{11}C_{44} \cdot \left\{ p^4 - \left(8\frac{C_{22}}{C_{44}} - 8C_{11}\frac{C_{12}}{C_{44}} - 8\frac{C_{12}}{C_{11}} + \frac{C_{22}}{C_{11}} + 1 \right) p^2 + 9\frac{C_{22}}{C_{11}} \right\}. \quad (25)$$

Complete Solution

Adding the homogeneous terms gives

$$\begin{aligned} u_r = & Ar^p + Br^{-p} - \frac{C_{12} - C_{22}}{C_{11} - C_{22}} Gr - \frac{C_{13} - C_{23}}{C_{11} - C_{22}} \epsilon_{zz} + \Delta u_{r0} \\ & + (\gamma_1 Cr^s + \gamma_2 Dr^s + \gamma_3 Er^s + \gamma_4 Fr^s + \Delta u_{r2}) \cos 2\theta, \quad (26) \end{aligned}$$

$$u_\theta = Gr\theta + (Cr^s + Dr^s + Er^s + Fr^s + \Delta u_{\theta 2}) \sin 2\theta, \quad (27)$$

where p is given by Eq. (18), s_k by Eq. (21), and γ_k by Eq. (22). In addition Δu_{r0} is the term independent of theta in Eq. (23), Δu_{r2} the coefficient of $\cos 2\theta$ in Eq. (23), and $\Delta u_{\theta 2}$ the coefficient of $\sin 2\theta$ in Eq. (24).

Application of Eqs. (8-11) gives the complete solution for the strains. Then the use of Eqs. (4-7) will give complete solutions for stresses. Note that there are seven unknowns

for each region of interest (ABGCDEF) and one unknown ϵ_{zz} that is common to all regions.

Boundary Conditions

In the present problem three cylindrical regions are employed. Region 1 is between $r=a$ and $r=b$ and is characterized by the elastic properties of a structural bore tube. If this is not present as in the Fermilab Doubler⁵ then it is removed by entering a very weak structure. Region 2 is between $r=b$ and $r=c$ and is used to characterize the region of superconductor. Region 3 is between $r=c$ and $r=d$ and is used to characterize the retaining collar.

Boundary conditions require surface traction and displacement to be continuous. One exception is allowed however. The notion of a dislocation or discontinuity in displacement is useful in characterizing pre-strain. Thus in detail:

$$\text{At } r=a, \quad \sigma_{rr}^{(+)} = \sigma_{r\theta}^{(+)} = 0, \quad (28)$$

$$\text{dislocation} \quad u_\theta^{(+)}(2\pi) - u_\theta^{(+)}(0) = a\alpha_1, \quad (29)$$

$$\text{At } r=b, \quad \sigma_{rr}^{(+)} - \sigma_{rr}^{(-)} = \sigma_{r\theta}^{(+)} - \sigma_{r\theta}^{(-)} = 0, \quad (30)$$

$$\text{dislocation} \quad u_\theta^{(+)}(2\pi) - u_\theta^{(+)}(0) = b\alpha_2, \quad (31)$$

$$u_r^{(+)} - u_r^{(-)} = u_\theta^{(+)} - u_\theta^{(-)} = 0, \quad (32)$$

$$\text{At } r=c, \quad \sigma_{rr}^{(+)} - \sigma_{rr}^{(-)} = \sigma_{r\theta}^{(+)} - \sigma_{r\theta}^{(-)} = 0; \quad (33)$$

$$\text{dislocation} \quad u_\theta^{(+)}(2\pi) - u_\theta^{(+)}(0) = c\alpha_3, \quad (34)$$

$$u_r^{(+)} - u_r^{(-)} = u_\theta^{(+)} - u_\theta^{(-)} = 0. \quad (35)$$

$$\text{At } r=d, \quad \sigma_{rr}^{(-)} = \sigma_{r\theta}^{(-)} = 0. \quad (36)$$

Application of these conditions provides 21 relations.

Integral Condition Satisfied by Stresses

The final condition needed to provide as many relations as unknowns may be obtained from the virial theorem⁶. This theorem of mean stress gives

$$\iint (\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) r dr d\theta = \frac{1}{3} \pi^2 J_0^2 \left\{ -\frac{1}{2}(c^4 - b^4) + [c + (c^3 - b^3)r_s^{-2}](c^3 - b^3) - b^3(c - b) \right\}, \quad (\text{dynes}) \quad (37)$$

where the RHS equals the magnetic energy per unit length plus a surface integral of Maxwell stresses on the iron.

Numerical Results

Stress, strain and displacement have been calculated for the Doubler⁷. For notation R is radial, T is theta or azimuthal, Z is axial or longitudinal, P and M designate positive and negative side of a given cylinder which is designated by A, B, C, D. Thus, for example, RTBP indicates the (r, θ) component on the positive side of the cylinder $r=b$. Reasonable estimates have been used for elastic and

thermal constants.⁶

References

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FLASTIC STRESS AND STRAIN IN ORTHOTROPIC STRUCTURE DOUBLER DIPOLE. Table with columns for parameters (e.g., CENTRAL MAGNETIC FIELD, OUTER CONDUCTOR RADIUS) and stress/strain components (e.g., TRANSVERSE STRESS, TRANSVERSE STRAIN, TRANSVERSE DISPLACEMENT).

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