IEEE Transactions on Nuclear Science, Vol. NS-26, No. 3, June 1979

MODELING A HORIZONTAL WIGGLER IN AN ELECTRON STORAGE RING*

Richard H. Helm

Stanford University, Stanford, California 94305

Summary

The effects of a wiggler on the beam parameters depend on several integrals involving the machine functions and the field distribution in the wiggler. It is shown that these integrals are separable into sums of products of terms containing only the initial values of the machine functions, and terms containing integrals over the wiggler fields. The field-dependent integrals may be determined by numerical integrations based on measured field distribution. In typical wiggler designs, the energy and excitation dependencies of the integrals may be modeled mathematically by simple power series.

Introduction

A wiggler is a system of bending magnets of alternating sign, placed in a storage ring to produce one or more of the following effects:

(1) to modify the damping rates;
(2) to modify the emittance of the beam;
(3) to decrease the polarization time; or
(4) to enhance the synchrotron radiation spectrum.

In flat wigglers,[1,2] which are considered here, the bends are all in the same plane. Helical wigglers[3,4] have been proposed and extensively studied in the literature, but will not be discussed in the present note.

For the purpose of establishing preliminary design criteria, a flat wiggler usually is treated as a number of "squared-off" or "hard-edge" magnets. However, in actual wigglers, much of the bending takes place in the end fields and therefore the beam effects, which depend on integrals of various powers of the local field, can not be expressed accurately by hard-edge models.

The purpose of the present note is to show how the various effects of the wiggler may be modeled in a simple way suitable for use in machine control. It will be seen that in general a total of about 17 functions are involved. However, in typical designs many of these functions vanish identically because of symmetries, and others are negligibly small. Furthermore, each of the functions may be modeled quite accurately by a simple power law in $(B_0/E)^n$ where B_0 is a measure of the field excitation, E is the beam energy, and n is an integer which takes on values of either 0, 2, 3, 4 or 5 for the different functions. Magnet saturation may cause the field distribution to vary with excitation so that the series coefficients would vary slowly with B_0.

A computer program has been used to obtain numerical results for typical wiggler designs. In practice the required functions could be determined either by computer analysis of the measured field data, or by experimental calibration using the stored beam in the ring.

Beam Optics

It will be assumed that the wiggler is to be inserted in a straight section. Two constraints then are imposed: (1) no net deflection of the beam; and (2) no net offset of the beam. These criteria are satisfied if the net field integral vanishes:

$$\int B\, ds = 0 \tag{1}$$

and if the bending field is symmetric about the midpoint. Fig. 1 illustrates coordinate nomenclature. Here X,Z are rectangular coordinates in the midplane, s is the distance along the design orbit, x is the horizontal coordinate normal to s, and $Y(Z) = y(s)$ is the vertical coordinate. The fields are assumed to have midplane

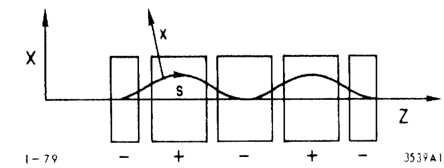

Figure 1. Wiggler Coordinate System

symmetry:

$$B_y(y) = B_y(-y) \tag{2}$$

and to be uniform in the X direction over the useful beam region:

$$\frac{\partial B_y}{\partial X} = 0 \tag{3}$$

The design orbit may be found by numerical integration of the equations of motion in the midplane (radiation reaction is neglected):

$$X'(Z) = p_X / p_Z \tag{4}$$

$$cp_X'(Z) = -e\,B \tag{5}$$

$$s'(Z) = \sqrt{1+X'^2} \tag{6}$$

$$E'(Z) = 0 \tag{7}$$

where

$$cp_Z = \sqrt{E^2 - (mc^2)^2 - c^2 p_X^2}$$

$$B = B_y(0,\ 0,\ Z) \text{ (the midplane field)}$$

and

$$X(0) = X'(0) = 0$$

The circumference of the ring is perturbed by an amount

$$\Delta L = \ell - \ell_0 = \int s'\, dZ - \ell_0 \tag{8}$$

where ℓ_0 is the length along the Z axis and ℓ is the length along the design orbit.

The transport matrix for linearized motion relative to the design orbit is defined by

$$x(s) = T(s)\ x(0) \tag{9}$$

where the components of x(s) are $[x, x', y, y', c\Delta t, \Delta E/E]$. (In the relativistic limit $c\Delta t = \Delta s =$ the path-length difference and $\Delta E/E = \Delta p/p =$ the relative momentum deviation). Because of the assumptions of midplane symmetry and constant energy, the x,y and y,E coupling terms all vanish. The non-trivial elements $t_{ij}(s)$ of the matrix T(s) are found by integration of the linearized equations of motion:

$$\left.\begin{aligned} t'_{1j} &= t_{2j} && (10)\\ t'_{2j} &= -(K+h^2)t_{1j} && (11)\end{aligned}\right\}\quad j = 1,2$$

$$t'_{16} = t_{26} \tag{12}$$

$$t'_{26} = -(K+h^2)t_{16} + h \tag{13}$$

$$\left.\begin{aligned} t'_{3j} &= t_{4j} && (14)\\ t'_{4j} &= Kt_{3j} && (15)\end{aligned}\right\}\quad j = 3,4$$

$$t'_{5j} = ht_{1j} \qquad j = 1,2,6 \tag{16}$$

where

$$h = 1/\rho = B/(B\rho)$$

$$K = 1/(B\rho)\,\partial B/\partial x$$

$$t_{ij}(0) = \delta_{ij}$$

*Work supported by the Department of Energy under Contract Number EY-76-C-03-0515.

0018-9499/79/0600-3824$00.75 © 1979 IEEE

As an alternative to integration of Eq. (16), t_{51} and t_{51} may be found from

$$t_{51} = t_{11}t_{26} - t_{21}t_{16} \quad (17)$$
$$t_{52} = t_{12}t_{26} - t_{22}t_{16}$$

which result from the symplectic property or may be shown directly from the equations of motion.

Let the matrix for the transformation all the way through the wiggler be defined by

$$M = (m_{ij}) = T(\ell) \quad (19)$$

Because of the constraints of zero deflection and zero offset, and the symplectic property, it follows that

$$m_{16} = m_{26} = m_{51} = m_{52} = 0 \quad (20)$$

Also, from the assumption that the fields are uniform in the X direction, it follows that

$$m_{21} = 0 \quad (21)$$

Thus the (3×3) horizontal matrix may be written

$$M_x = \begin{bmatrix} m_{11} & m_{12} & m_{16} \\ m_{21} & m_{22} & m_{26} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \ell_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (22)$$

where

$$\ell_x = m_{12} \quad (23)$$

In the vertical, $m_{33} = m_{44}$ by symmetry, and we may write

$$M_y = \begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} 1 & \tfrac{1}{2}\ell_y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_y & 1 \end{bmatrix} \begin{bmatrix} 1 & \tfrac{1}{2}\ell_y \\ 0 & 1 \end{bmatrix} \quad (24)$$

where

$$k_y = -m_{43} \quad (25)$$

$$\ell_y = \frac{2}{k_y}(1 - m_{33}) \quad (26)$$

Thus the optics are described by the four functions ℓ, ℓ_x, ℓ_y and k_y given by the equations (7), (23), (25), and (26). Modeling of these functions for use in machine control will be discussed in a later section.

Synchrotron Radiation Integrals

In order to describe the synchrotron motion, energy loss, damping, energy spread and beam emittance, we need the following five integrals:[5,6]

$$I_1 = \oint h\eta ds \quad (27)$$
$$I_2 = \oint h^2 ds \quad (28)$$
$$I_3 = \oint |h^3| ds \quad (29)$$
$$I_4 = \oint (h^3 + 2hK)\eta ds \quad (30)$$
$$I_5 = \oint |h^3| H ds \quad (31)$$

with

$$H = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2 \quad (32)$$

where γ, α, β, η and η' are the Courant and Snyder machine functions[7] in the horizontal plane (the subscript x is omitted for the sake of brevity). The relationship of these integrals to the machine parameters is given in Appendix A.

The contributions of the wiggler to the above integrals may be expressed in terms of the initial values of the Courant-Snyder functions, and integrals involving only the properties of the wiggler itself. As will be shown in Appendix B,

$$\Delta I_1 = \eta_0 \Delta I_{11} + \eta_0' \Delta I_{12} + \Delta I_{16} \quad (33)$$
$$\Delta I_2 = \int h^2 ds \quad (34)$$
$$\Delta I_3 = \int |h^3| ds \quad (35)$$
$$\Delta I_4 = \eta_0 \Delta I_{41} + \eta_0' \Delta I_{42} + \Delta I_{46} \quad (36)$$
$$\Delta I_5 = H_0 \Delta I_3 + (\alpha_0\eta_0 + \beta_0\eta_0')\Delta I_{51} + (\gamma_0\eta_0 + \alpha_0\eta_0')\Delta I_{52} + \beta_0 \Delta I_{511} + \alpha_0 \Delta I_{512} + \gamma_0 \Delta I_{522} \quad (37)$$

where

$$H_0 = \gamma_0\eta_0^2 + 2\alpha_0\eta_0\eta_0' + \beta_0\eta_0'^2 \quad (38)$$
$$\Delta I_{1j} = \int h t_{1j} ds = m_{5j}, \quad j = 1,2,6 \quad (39)$$
$$\Delta I_{4j} = \int (h^3 + 2hK) t_{1j} ds, \quad j = 1,2,6 \quad (40)$$
$$\Delta I_{51} = 2\int |h^3| t_{51} ds \quad (41)$$
$$\Delta I_{52} = -2\int |h^3| t_{52} ds \quad (42)$$
$$\Delta I_{511} = \int |h^3| t_{51}^2 ds \quad (43)$$
$$\Delta I_{512} = -2\int |h^3| t_{51} t_{52} ds \quad (44)$$
$$\Delta I_{522} = \int |h^3| t_{52}^2 ds \quad (45)$$

Modeling the Wiggler

Obviously a machine control program is not going to evaluate on-line all the numerical integrals outlined above. What is needed is a simple mathematical model of each function, such as power series in $h_0 = eB_0/E$ (where B_0 is the wiggler field, say at the symmetry point). The coefficients of the power series could be precalculated from magnetic measurement data or found by experiments with stored beams.

Fortunately several of the functions defined above vanish identically because of symmetries and other properties of the wiggler. The functions which vanish are:

m_{21}, m_{16} and m_{26} (horizontal focusing and dispersion);

$\Delta I_{11} = m_{51}$ and $\Delta I_{12} = m_{52}$ (path length terms);

ΔI_{41} and ΔI_{42} (damping terms);

ΔI_{51} and ΔI_{512} (quantum excitation terms).

The remaining functions for a given wiggler field distribution may be represented by power series of the form

$$f = h_0^n (F_0 + F_1 h_0^2 + F_2 h_0^4 + \ldots) \quad (49)$$

Table I, below, gives the functional dependence of the leading terms on h as well as on the wiggle period λ:

Integral	Leading Term in Expansion
ΔL, $\Delta\ell_x$, $\Delta\ell_y$, ΔI_{16}	$h_0^2\lambda^3$
k_y	$h_0^2\lambda$
ΔI_2	$h_0^2\lambda$
ΔI_3	$\lvert h_0^3\rvert\lambda$
ΔI_{46}, ΔI_{52}	$h_0^4\lambda^3$
ΔI_{511}	$\lvert h_0^5\rvert\lambda^3$
ΔI_{522}	$\lvert h_0^5\rvert\lambda^5$

TABLE I

Functional Dependence of Leading Term of Expansions of Wiggler Functions. A factor of N (Number of Wiggle Periods) is implied for each function.

where $\Delta L = \ell - \ell_0$, $\Delta\ell_x = \ell_x - \ell_0$, $\Delta\ell_y = \ell_y - \ell_0$. The results in Table I have been found by computer experiments. Another useful result is $k_y = \Delta I_2$ in the lowest order. If the field distribution changes appreciably because of saturation effects, the F_j may be slowly varying functions which could be represented as power series in B_0. Numerical examples for SPEAR and PEP wigglers are given elsewhere.[8,9,10] Comparison of calculated results to experiments in SPEAR are given by Berndt, et al.[8] For normal-conducting wiggler magnets, it appears that only the terms k_y, ΔI_2, ΔI_3 and (possibly) ΔI_{511} need be considered.

References

1. E. M. Purcell, "Production of Synchrotron Radiation by Wiggler Magnets" (1972), unpublished; see also Stanford Linear Accelerator Center SSBP Report 77/05 (1977), eds. H. Winick and T. Knight for this note and many other reports on wigglers.
2. J. M. Paterson, J. R. Rees and H. Wiedemann, "Control of Beam Size and Polarization Time in Pep", Stanford Linear Accelerator Center Report PEP-125 (1975).
3. B. M. Kincaid, "A Short Period Helical Wiggler as an Improved Source of Synchrotron Radiation", J. Appl. Phys. 48, 7, pp 2684-91 (1977).
4. J. P. Blewett and R. Chasman, "Orbits and Fields in Helical Wigglers", J. Appl. Phys. 48, 7, pp 2692-8 (1977).
5. M. Sands, "The Physics of Electron Storage Rings, An Introduction", Proceedings of the International School of Physics "Enrico Fermi", Course XLVI, ed., B. Touschek, Academic Press (1971); also Stanford Linear Accelerator Report SLAC-121 (1970).
6. R. H. Helm, M. J. Lee, P. L. Morton and M. Sands, "Evaluation of Synchrotron Radiation Integrals", IEEE Trans. on Nuc. Sci., NS-20, 3 p 900 (1973).
7. E. D. Courant and H. S. Snyder, "Theory of the Alternating Gradient Synchrotron", Ann. Of Phys. 3, p 1-48 (1958).
8. M. Berndt, W. Brunk, R. Cronin, D. Jensen, R. Johnson, A. King, J. Spencer, T. Taylor and H. Winick, "Initial Operation of SSRL Wiggler in SPEAR", this conference.
9. W. Brunk, G. Fischer and J. Spencer, "Single Wavelength Standard Wiggler for PEP", this conference.
10. R. H. Helm, "Modeling the Effects of a Flat Wiggler on a Storage Beam," Stanford Linear Accelerator Report PEP-272 (1978).

Appendix A. Beam Parameters

The integrals I_1, I_2, I_3, I_4 and I_5 (Eqs. 33 - 37) are related to beam parameters as follows[5,6]

$$\alpha_p = I_1 / L \tag{A1}$$

$$U_0 = \frac{2}{3} r_e \frac{E^4}{(mc^2)^3} I_2 \tag{A2}$$

$$\alpha_y = \frac{1}{3} r_e \left(\frac{E}{mc^2}\right)^3 \frac{c}{L} I_2 \tag{A3}$$

$$\alpha_x = \alpha_y (1 - I_4 / I_2) \tag{A4}$$

$$\alpha_E = \alpha_y (2 + I_4 / I_2) \tag{A5}$$

$$\left(\frac{\sigma_E}{E}\right)^2 = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \left(\frac{E}{mc^2}\right)^2 \frac{I_3}{2I_2 + I_4} \tag{A6}$$

$$\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \left(\frac{E}{mc^2}\right)^2 \frac{I_5}{I_2 - I_4} \tag{A7}$$

where

α_p = (E/L) dL/dE = "momemtum compaction";

L = ring circumference;

U_0 = energy loss per turn;

$r_e = e^2/mc^2$ = classical electron radius;

α_x, α_y, α_E are the damping rates in x, y and E;

$\hbar/mc$ is the reduced Compton wavelength of an electron;

σ_E is the rms energy width;

and

ε_x is the horizontal beam emittance, in the absence of x-y coupling.

Appendix B. Effect of Wiggles on the Integrals

The transformation of $\vec{\eta}$ to any point in the wiggler may be written

$$\vec{\eta} = T_x \vec{\eta}_0 + d \tag{B1}$$

where

$$\vec{\eta} = \begin{bmatrix} \eta \\ \eta' \end{bmatrix}, \quad T_x = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \text{ and } d = \begin{bmatrix} t_{16} \\ t_{26} \end{bmatrix}$$

To find how the function H transforms, we note that

$$H = \tilde{\vec{\eta}} B \vec{\eta} \tag{B2}$$

where

$$B = \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} \text{ and } \tilde{\vec{\eta}} \text{ is the transpose of } \vec{\eta}.$$

The matrix B is related to the invariant function W:

$$W = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \tilde{\vec{x}} B \vec{x}$$

from which it follows that the transformation of B is

$$B = \tilde{T}_x^{-1} B_0 T_x^{-1} \tag{B3}$$

By combining Eqs. (B3), (B2), (B1) and using (17) and (18), we find

$$\begin{aligned} H &= \tilde{\vec{\eta}}_0 B_0 \vec{\eta} + 2\tilde{\vec{\eta}}_0 B_0 T_x^{-1} d + \tilde{d}\, \tilde{T}_x^{-1} B_0 T_x^{-1} d \\ &= H_0 + 2(\alpha_0\eta_0 + \beta_0\eta_0')t_{51} - 2(\gamma_0\eta_0 + \alpha_0\eta_0')t_{52} \\ &\quad + \beta_0 t_{51}^2 - 2\alpha_0 t_{51} t_{52} + \gamma_0 t_{52}^2 \end{aligned} \tag{B4}$$

Finally, by inserting Eqs. (B1) and (B4) in (27) through (31), we find Eqs. (33) through (45).