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COMPUTATIONAL METHOD FOR THE DISPERSION, BETATRON FUNCTIONS AND C.O.D. IN NONLINEAR LATTICE

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Summary

A computational method for the dispersion, betatron functions and the closed orbit distortion in the lattice with nonlinear elements is presented. We apply the method to the lattice with sextupole elements and with magnetic imperfections of KEK PHOTON FACTORY STORAGE RING (PF-RING), and present the numerical results.

I. Introduction

In accelerators and storage rings, the sextupole magnets are usually installed to compensate the natural chromaticity. In the lattice with nonlinear elements such as sextupole magnets, the linear dynamics of particle is disturbed by the nonlinear field, and the higher-order effects of the momentum deviation $\Delta p/p$ become important.[1] If we restrict ourselves to the oscillation with a small amplitude, the motion of particles in the nonlinear lattice is characterized by the dispersion and betatron functions and the closed orbit distortion, similar to the case of the linear lattice. The method of computing the dispersion and betatron functions assuming no field errors and no coupling between horizontal and vertical betatron oscillations has been studied previously,[2] the outline of which will be described in Sec.II. We have developed the computational method of the closed orbit distortion (C.O.D.) and orbit parameters including field errors and coupling effects. Sec.III will describe the computational method as well as the method of the corrections of C.O.D., η_y and coupling. Finally we will present the numerical examples in Sec.IV.

II. Momentum dependent dispersion and betatron functions without field errors[2]

Our aim is at first to obtain the equilibrium orbit of off-momentum particles or the dispersion function, and then to calculate the betatron function around that orbit. The computational steps are as follows; (i) to make the linearized transfer matrix for each nonlinear element in the vicinity of the initial guess of the equilibrium orbit, (ii) to calculate the periodic solution of particle motion from the linearized transfer matrix for one revolution, and correct the initial guess of dispersion function, (iii) subsequently to iterate such procedures until a sufficient convergence is obtained, (iv) finally to calculate the betatron function from the last linearized matrix. The generalized procedures of obtaining the linearized matrix with nonlinear element start from the equation of the transverse motion of a particle in the nonlinear element,

$$x'' + f(x, \Delta p/p) = 0, \tag{1}$$

where f is the nonlinear function of x and $\Delta p/p$. The formal solution g of Eq.1 will be given as follows;

$$\begin{pmatrix} x_e \\ x'_e \\ 1 \end{pmatrix} = \begin{pmatrix} g(x_i, x'_i, \Delta p/p) \\ g'(x_i, x'_i, \Delta p/p) \\ 1 \end{pmatrix} \tag{2}$$

where i and e denote the entrance and exit of the element. The first-order correction $\Delta\vec{x}$ is obtained by linearizing the function g,

$$\begin{pmatrix} \Delta x_e \\ \Delta x'_e \\ 1 \end{pmatrix} = \begin{pmatrix} \partial g/\partial x, & \partial g/\partial x', & g-x_e \\ \partial g'/\partial x, & \partial g'/\partial x', & g'-x'_e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_i \\ \Delta x'_i \\ 1 \end{pmatrix} . \tag{3}$$

For example, the linearized matrix of transforming the horizontal motion in the sextupole magnet that is assumed to be a thin lens, is given by,

$$\begin{pmatrix} \Delta x_e \\ \Delta x'_e \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2\lambda x_{eq} & 1 & -\lambda x_{eq}^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_i \\ \Delta x'_i \\ 1 \end{pmatrix} \tag{4}$$

at the first iteration. Here x_{eq} is the equilibrium orbit of off-momentum particle, i.e. $x_{eq} = \eta\Delta p/p$. The (2,3)-component of the matrix should be replaced by $-\lambda(\Delta x_{eq})$ from the second iteration. The linearized matrices of other elements are given in the Ref.2.

The degree of convergence of the iteration process is given by

$$x_{eq} = \sum_{n=0}^{\infty} \eta_n (\Delta p/p)^{2^n} ,$$

where η_n is the n-th coefficient of the series and depends on $\Delta p/p$. It is easily seen that this series converges very fast. Practically, four iterations are sufficient to compute the dispersion function for $\Delta p/p$ = 1 % with the convergence criterion of $|\Delta x_{eq}/\Delta p/p| = 10^{-10}$ m. The series of the iteration for the computation of C.O.D. described in Sec.III gives a similar convergence speed.

III. C.O.D., dispersion, coupling and their corrections[3]

In the following, will be explained the computational method for the C.O.D. due to the field errors and misalignments in the lattice with nonlinear elements, the dispersion around the C.O.D., and the coupled betatron functions around the C.O.D. The procedures of the computations for the momentum-dependent dispersion and the momentum-dependent betatron functions around the equilibrium orbit of off-momentum particles will be also explained. The details of the computational method will be published elsewhere.[3]

C.O.D.

The C.O.D. $\vec{x}_c$ including coupling due to the magnetic imperfections in the lattice with nonlinear element can be calculated by a method quite similar to that in Sec.II. choosing the C.O.D. without nonlinear magnets as initial guess. The linearized equation of C.O.D. has been solved by the Gauss elimination method.

Dispersion

The dispersion around the C.O.D. can be also calculated by the same method as that of C.O.D. Only the transfer matrix of dispersion η in the quadrupole element with misalignment δx is given as an example, because of its significant effect to the distortion of η. The C.O.D. x_c and equilibrium orbit x_{eq} of off-momentum

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particle in an unrotated quadrupole magnet are expressed by,

$$x''_c + Kx_c = K\delta x \qquad x''_{eq} + K/(1+\Delta p/p)x_{eq} = K\delta x/(1+\Delta p/p).$$

Since the dispersion is given by,

$$\vec{\eta} = \lim_{\Delta p/p\to 0} (\vec{x}_{eq} - \vec{x}_c)/\Delta p/p ,$$

the following matrix representation is obtained,

$$\vec{\eta}^e = M_0\vec{\eta}^i + M'_0\vec{x}^i_c + \vec{b}'_0$$

where

$$M'_0 = \partial M(\Delta p/p)/\Delta p/p|_{\Delta p/p=0} , \ \vec{b}'_0 = \partial\vec{b}(\Delta p/p)/\Delta p/p|_{\Delta p/p=0}.$$

M_0 and $M(\Delta p/p)$ denote the usual transfer matrices without errors of right momentum and off-momentum particles, respectively, and $\vec{b}$ is the vector due to the misalignment and is given by $\vec{b} = (I-M)\delta\vec{x}$. The transfer matrix for a rotated quadrupole magnet is easily obtained by replacing M and $\vec{b}$ by $M_\theta^{-1}MM_\theta$ and $\delta\vec{x}-M_\theta^{-1}MM_\theta\delta\vec{x}$, where M_θ is the rotation matrix. Putting the above obtained dispersion as initial guess, the momentum-dependent dispersion can be computed in the similar manner as in Sec.II. The transfer matrix of the momentum-dependent dispersion in a quadrupole magnet is represented by,

$$\vec{\eta}^e = M(\Delta p/p)\eta^i + (M(\Delta p/p)\vec{x}^i_c+\vec{b}(\Delta p/p)-\vec{x}^e_c)/\Delta p/p.$$

The nonzero off-diagonal elements of the linearized matrix for the sextupole magnet are given as follows;

$$m_{21} = -m_{43} = \frac{-2\lambda(x_c+\eta_x\Delta p/p)}{1+\Delta p/p} \quad , m_{23} = m_{41} = \frac{2\lambda(y_c+\eta_y\Delta p/p)}{1+\Delta p/p}$$

and at first iteration m_{25} and m_{45} are,

$$m_{25} = \frac{-\lambda}{1+\Delta p/p} \left[\eta_x^2-\eta_y^2-2(\eta_x x_c-\eta_y y_c)+(x_c^2-y_c^2)\right]\Delta p/p$$

$$m_{45} = \frac{2\lambda}{1+\Delta p/p} \left[\eta_x\eta_y-(\eta_y x_c+\eta_x y_c)+x_c y_c\right]\Delta p/p,$$

and from second iteration,

$$m_{25} = \frac{-\lambda}{1+\Delta p/p}[(\Delta\eta_x)^2-(\Delta\eta_y)^2]\Delta p/p, \ m_{45} = \frac{2\lambda}{1+\Delta p/p}\Delta\eta_x\Delta\eta_y\Delta p/p.$$

Coupled betatron oscillation

The coupled betatron oscillations and the beam parameters - damping time, beam size etc. - have been extensively analyzed by Ripken.[4,5] To compute the eigenvectors and eigenvalues of betatron oscillation, we use the symplectic rotation.[6] Computing the eigenvectors at one position, we can easily obtain those at the other places by multiplying the transfer matrices. Note here that the transfer matrix of the canonical variables for one revolution equals to that of the coordinates and their derivatives at the location without the longitudinal magnetic fields, even if the fields exist in other places around the ring. The computational steps for the eigenvectors are as follows; (i) to decompose the transfer matrix into

$$T=RUR^{-1} \quad , \quad R=\begin{pmatrix} I\cos\theta, & D^{-1}\sin\theta \\ -D\sin\theta, & I\cos\theta \end{pmatrix}, \quad U=\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where the notations are those of Edwards and Teng,[6] and A and B are in the familiar form of Courant-Snyder, (ii) to calculate the symplectic normalized eigenvectors of A and B, (iii) and then to multiply the eigenvectors by R to obtain the eigenvectors of T. The method explained here would be similar to the computer code PETROS.[7] To simplify the computation of beam parameters, we rotate the co-ordinate systems in the quadrupole magnets with rotated errors by the same angles. The chromaticity is analytically calculated in each element, and the other beam parameters are numerically integrated by Simpson's formula and Richardson's extraporation, dividing each element into forty subsections.

Correction of C.O.D., vertical dispersion

In this report, we use the least squares method[8,9] for the correction of C.O.D. and η_y, but use the correction matrix of thick lenses. The correction matrix is formally obtained from the transformation G of $\vec{x}_c$ for one revolution as follows;

$$\vec{x}_c = G(\vec{x}_c, \vec{s}) , \qquad (5)$$

where $\vec{s}$ is the vector of strengths of the correctors. From Eq.5, the first-order correction of C.O.D. by the correctors is given by,

$$\Delta\vec{x}_c = (I - \partial G/\partial\vec{x})^{-1}\partial G/\partial\vec{s}\ \Delta\vec{s}. \qquad (6)$$

We make the correction matrix from the first and third rows of the matrix of Eq.6. Practically, we cannot but use the linear transformation for G. Similarly, we obtain the η_y-correction matrix from

$$\Delta\vec{\eta} = (I-\partial H/\partial\vec{\eta})^{-1}[\partial H/\partial\vec{x}(I-\partial G/\partial\vec{x})^{-1}\partial G/\partial\vec{s}+\partial H/\partial\vec{s}]\Delta\vec{s}, \quad (7)$$

where H is the transformation of η for one revolution. The first and second terms of the bracket in the right-hand side of Eq.7 denote the effects of indirect and direct corrections, respectively. The first term contains the effects of C.O.D. in quadrupole magnets and those of C.O.D. and η_x in sextupole magnets.[10]

Coupling control

The betatron coupling will be corrected by setting skew quadrupole magnets in the lattice and using the method of Guignard's coupling coefficient.[11] Here, the measurement method of the betatron coupling are not described, but the method and possibility of the coupling control are explained. Following Guignard, we can write the equation of the coupling control as follows;

$$\Sigma a_i x_i + \mathrm{Re}\kappa = C\cos\theta, \quad \Sigma b_i x_i + \mathrm{Im}\kappa = C\sin\theta, \qquad (8)$$

where x_i is the strength of the i-th correction skew quadrupole magnet, κ the coupling coefficient due to rotated errors of quadrupole magnets, C,θ the modulus and phase of the desired coupling coefficient, and a_i, b_i the real and imaginary parts of the coefficient at the i-th skew quadrupole magnet with the strength of unity. The contribution to the coupling coefficient due to C.O.D. in the sextupole magnets is neglected here. We can solve Eq.9 in the form $\vec{x} = s\vec{a} + t\vec{b}$, choosing θ so as to minimize the Euclidean norm of the vector of skew quadrupole strengths.

IV. Numerical examples

The preliminary results computed by the method in Sections II and III are shown in Figures for the lattice of KEK PF-RING. The following r.m.s. values of errors are assumed; relative field error in bending magnets = 0.1 %, tilt error in bending magnets = 0.2 mrad, rotated quadrupole error = 0.2 mrad, horizontal and vertical misalignments of quadrupole magnets = 0.1 mm, position measurement error σ_{PM} = 0.2 mm, kick error in vertical corrections σ_{VD} = 0.01 mrad. We have tentatively assumed that the field gradient error in quadrupole magnet is zero. The σ in Fig.1 to Fig.3 denotes the unbiased standard deviation over 100 'machines'. The 'machine with errors' in Fig.4 to

Fig.8 denotes the case with the largest $x_{C.O.D.}$ among 100 'machines'. Here the correction of C.O.D. has not been iterated, and the re-correction of the chromaticity by sextupole magnets has not also been made. For the correction of η_y, it is assumed that the measurement error of η_y is zero. This may be an impractical assumption for PF-RING since the measurement errors of η_y would be comparable to the small η_y itself after the C.O.D. correction. In Figs. 7 and 8, the nominal tunes have been shifted perpendicular to the line of difference resonance in the tune diagram, by changing the strengths of the quadrupole magnets in the insertions. The remaining coupling and tune shift after the coupling correction (κ=C=0) as seen in Figs.6 and 8 is due to the vertical C.O.D. in the sextupole magnets. The other parameters not shown in Figures - r.m.s. values of C.O.D. and the distortion of dispersion at the positions of all elements, damping times and beam envelopes etc. - have also been computed.

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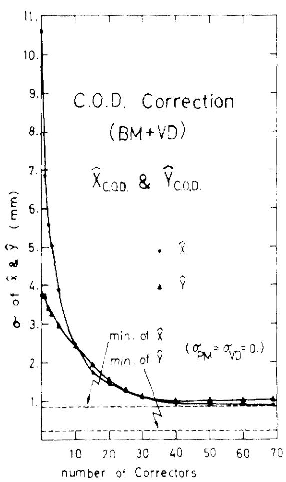

Fig.1

The standard deviations of the maximum C.O.D. versus the number of correctors.

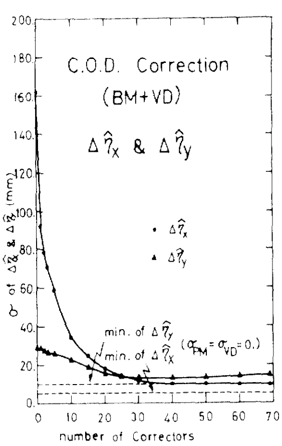

Fig.2

The standard deviations of the maximum distortion of dispersion in the C.O.D. correction.

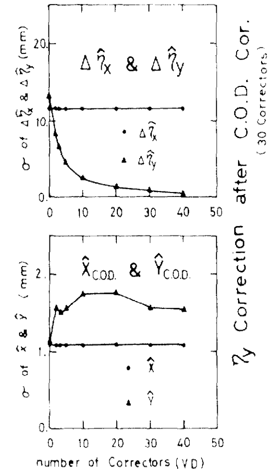

Fig.3

The variations of dispersion and C.O.D. in the vertical dispersion correction after the C.O.D. has been corrected by 30 correctors.

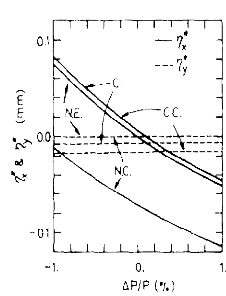

Fig.4

The variations of dispersion versus the momentum deviation Δp/p. The suffix * denotes the center of the insertion. N.E., N.C., C. and C.C. denote the cases without errors, without C.O.D. correction, with C.O.D. correction (by 40 correctors), and with both C.O.D. correction and coupling compensation (by four skew quadrupole magnets), respectively.

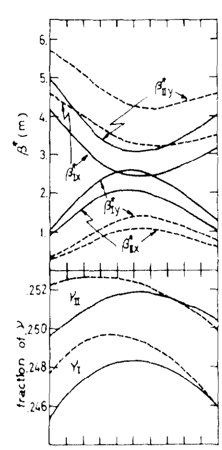

Fig.5

β and ν versus Δp/p. The full and broken lines denote the cases with and without C.O.D. correction, respectively.

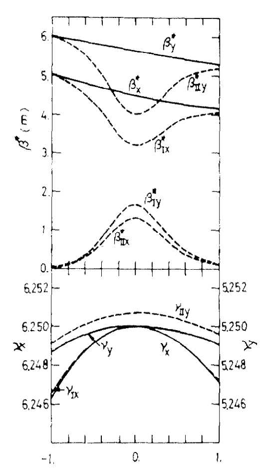

Fig.6

β and ν versus Δp/p. The full and broken lines denote the cases without errors, and with both C.O.D. correction and coupling compensation, respectively.

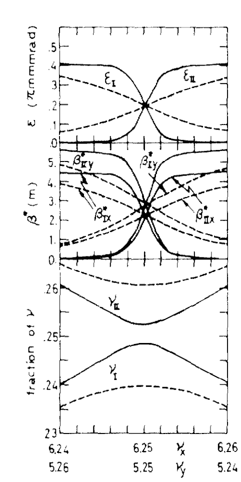

Fig.7

β, ν and emittance versus nominal tunes. The full and broken lines denote the cases with C.O.D. correction and with both C.O.D. correction and coupling control (κ = 0, C = 0.01), respectively.

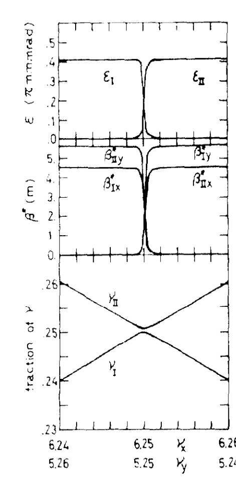

Fig.8

β, ν and emittance versus nominal tunes for the case with both C.O.D. correction and coupling compensation.