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SOME ASPECTS ON LINEAR AND NON-LINEAR ORBIT MOTION IN STORAGE RINGS.

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Summary.

Our purpose is the development of a general analytical theory which describes the orbit motion of particles in a storage ring, including the effect of the acceleration, analogous as done for cyclotrons[1,2]. We only present here the start, neglecting the effect of the RF system.

We will use a Hamilton formalism with action and angle variables and show that this leads, after several correct transformations[2,3], to expressions for the off-energy function , tunes and chromaticities. This theory can be extended to the influence of non-linear elements as sextupoles and octupoles and long straight sections on the orbit motion.

Our treatment only needs *direct field quantities* in the form of Fourier components of the guide field elements. No *a priori* information about the lattice functions (η,β etc.) is required, whereas normally one uses[4,5] the behaviour of these functions as provided by lattice designing matrix codes.

The results of the analytical theory as given here will be compared with numerical results applied on the lattice of the proposed Dutch dedicated synchrotron radiation storage ring PAMPUS[6].

Introduction.

The orbit motion of a charged particle in a magnetic field with respect to an equilibrium orbit, can be described in curvilinear co-ordinates by a general Hamiltonian H:

$$H = -R\left(1+\varepsilon\frac{x}{\rho}\right)\left\{\sqrt{p^2-p_x^2-p_z^2}-eA_\theta\right\} \qquad (1)$$

where: θ is the independent variable (azimuth),
ρ is the radius of curvature in the magnets,
R is the mean radius of the machine,
$\varepsilon=1$ inside the bending magnets, elsewhere $\varepsilon=0$.

Introducing reduced variables (i.e. $\bar{x} = x/\rho$, $p_x = \frac{d\bar{x}}{d\theta}$) and evaluating the square root in (1) up to the second degree of these variables lead to a more handsome Hamiltonian. For the vectorpotential A_θ, up to sextupole components, we use:

$$A_\theta = B_o\rho\left\{\varepsilon + \tfrac{1}{2}(\varepsilon-n)\bar{x}^2+\tfrac{1}{2}n\bar{z}^2 - \left(\frac{1}{6}S+\tfrac{1}{2}\varepsilon-\frac{1}{6}\varepsilon n\right)\bar{x}^3+\tfrac{1}{2}S\bar{x}\bar{z}^2\right\} \qquad (2)$$

where B_o is the fieldstrength in the bending magnets,

$$\text{and}\quad n = -\frac{\rho}{B_o}\left(\frac{\partial B_z}{\partial x}\right)_o,\quad S = -\frac{\rho^2}{B_o}\left(\frac{\partial^2 B_z}{\partial x^2}\right)_o \qquad (3)$$

are the quadrupole and sextupole components.

Describing the motion of a particle with a relative momentum deviation δ with respect to the central particle $p = p_c(1+\delta) = B_o\rho e(1+\delta)$, the Hamiltonian H of (1) now becomes, up to the third degree in the canonical variables and up to the first degree in δ:

$$\begin{aligned}\bar{H} = &-\varepsilon\delta\frac{R^2}{\rho^2}\bar{x}+\tfrac{1}{2}(1-\delta)\frac{R^2}{\rho^2}\{(\varepsilon-n)\bar{x}^2+n\bar{z}^2\} + \\ &+\tfrac{1}{2}(1+\varepsilon\bar{x})(\bar{p}_x^2+\bar{p}_z^2)-\frac{1}{6}(1-\delta)\frac{R^2}{\rho^2}S\{\bar{x}^3-3\bar{x}\bar{z}^2\} \\ &-\frac{1}{6}(1-\delta)\frac{R^2}{\rho^2}\varepsilon n\{2\bar{x}^3-3\bar{x}\bar{z}^2\}\end{aligned} \qquad (4)$$

We now introduce new variables, which are deviations with respect to the reference particle ($\bar{x}_o$, $\bar{p}_o$) with momentum deviation δ in the medium plane, generated by:

$$G(\bar{x},\pi_x,\bar{z},\bar{\bar{p}}_z) = \pi_x\bar{x}-\pi_x\bar{x}_o+\bar{x}\bar{p}_o+\bar{z}\bar{\bar{p}}_z \qquad (5)$$

The relations between the old and new variables are:

$$\pi_x = \bar{p}_x-\bar{p}_o;\ \xi = \bar{x}-\bar{x}_o;\ \bar{\bar{p}}_z = \bar{p}_z \text{ and } \bar{\bar{z}} = \bar{z}. \qquad (6)$$

From now on, we delete the bars above the variables.

After the transformation (5), the Hamiltonian can be written as a power expansion in the variables:

$$H(\xi,\pi_x,z,p_z) = H^{(1)}+H_u^{(2)}+H_p^{(2)}+H_u^{(3)}+\ldots \qquad (7)$$

where u and p means 'unperturbed' (δ=0) and 'perturbed'.

This Hamiltonian shows us the oscillations of the particles around (x_o,p_o), which is the closed orbit of the reference particle with momentum deviation δ. The central particle is characterized by $(x_o,p_o)=(0,0)$. The separate parts of this Hamiltonian will be treated in subjoined sections in order to deduce expressions for characteristic lattice properties.

Off-energy function.

In order to describe the betatronoscillations around the central orbit (x_o,p_o), the first degree parts in δ of the Hamiltonian (7) have to vanish: (8)

$$H^{(1)} = \pi_x\left\{p_o-\frac{dx_o}{d\theta}\right\}+\xi\left\{\frac{dp_o}{d\theta}+\frac{R^2}{\rho^2}(\varepsilon-n)x_o-\frac{R^2}{\rho^2}\delta\varepsilon\right\}=0$$

Transforming (8) to the independent variable s (ds=R.dθ) will lead to the differential equation:

$$\frac{d^2x_1}{ds^2} = \left\{-\frac{\varepsilon}{\rho^2}+\frac{n}{\rho^2}\right\}x_1+\frac{\varepsilon}{\rho^2} \qquad (9)$$

with $x_o = \delta.x_1$; $p_o = \delta.p_1$ and $p_1 = \frac{dx_1}{d\theta}$

Note that this differential equation just decribes the behaviour of the off-energy function $\eta(s) = \rho.x_1(s)$. Equation (9) can be solved by substituting the Fourier-components of the guide field[7].

Betatronnumbers.

In this section we will consider the unperturbed linear betatronmotion, in order to deduce analytical expressions for the betatronnumbers:

$$H_u^{(2)} = \tfrac{1}{2}\pi_x^2+\tfrac{1}{2}\frac{R^2}{\rho^2}(\varepsilon-n)\xi^2+\tfrac{1}{2}p_z^2+\tfrac{1}{2}\frac{R^2}{\rho^2}nz^2 \qquad (10)$$

We neglect coupling between the radial and vertical motions, so we can treat both modes independently. We will consider here only the radial motion in detail.

In dealing with a problem represented by (10) it is convenient to use action and angle variables[2] (I,ϕ):

$$\pi_x = \sqrt{2IQ_1}.\sin(\phi-Q_1\theta) \quad\text{and}\quad \xi = \sqrt{2I/Q_1}.\cos(\phi-Q_1\theta). \qquad (11)$$

This means that we study the radial motion in a phase plane rotating with a *fixed* frequency Q_1. The new Hamiltonian now becomes:

$$K(I,\phi) = H_u^{(2)}(\pi_x,\xi)-Q_1.I = e_2.I+f_2.I \qquad (12)$$

with e_2 the constant and f_2 the oscillating part.
We are looking for a transformation which removes the oscillating parts of (12). This can be generated by[2] :

$$G(\bar{I},\phi,\theta) = -\bar{I}.\phi-\bar{I}.U_2(\theta,\phi). \qquad (13)$$

After this transformation the new Hamiltonian becomes:

$$\bar{K} = \left\{e_2+f_2+e_2\left(\frac{\partial U_2}{\partial\phi}\right)+f_2\left(\frac{\partial U_2}{\partial\phi}\right)-\frac{\partial U_2}{\partial\theta}\right\}.\bar{I} \qquad (14)$$

The function U_2 is now determined by the fact that all oscillating parts in this Hamiltonian have to vanish, leading to the requirement:

$$f_2+e_2\left(\frac{\partial U_2}{\partial\phi}\right)+\mathrm{osc}\left(f_2.\frac{\partial U_2}{\partial\phi}\right)-\left(\frac{\partial U_2}{\partial\theta}\right) = 0. \qquad (15)$$

We keep the constant part, so that:

$$\bar{K} = e_2.\bar{I}+\left\langle f_2.\frac{\partial U_2}{\partial\phi}\right\rangle.\bar{I}, \qquad (16)$$

where the last term contains the influence of the field-modulation.

From (16) we now get an expression for the frequency of the betatronmotion, the tune or <u>betatronnumber</u> Q_x:

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$$Q_x = Q_1 + e_2 + \langle f_2 . \frac{\partial U_2}{\partial \phi} \rangle \tag{17}$$

Using in practice Q_1 as an estimation of the tune Q_x, we can find the correction term by evaluating (17). This will lead to an exact result if U_2 is correctly determined from (15).

It can be noted here that the relation between $\bar{I}$ and I given by (13) just contains the information about the behaviour of the beta-function $\beta(s)$[8].

We will illustrate this principle on the radial motion which is described by the radial part of (10). We expand the magnetic field in a Fourier series:

$$\frac{R^2}{\rho^2}(\varepsilon-n) = \tfrac{1}{2} A_o + \sum_n (A_n \cos n\theta + B_n \sin n\theta) = \omega^2 + \sum_n \gamma_n \cos n(\theta-\theta_n). \tag{18}$$

with n = k.N, k=1,2.. and N the (super)periodicity. Applying the transfromation to action and angle variables as given above and substituting (18), lead to the following expressions:

$$e_2 = -\frac{\Delta}{2} \quad \text{with } \Delta = Q_1 - \frac{\omega^2}{Q_1} \tag{19}$$

$$f_2 = -\frac{\Delta}{2} \cos(2\phi-2Q_1(\theta-\theta_n)) + \sum_n \frac{\gamma_n}{2Q_1} . \cos(n(\theta-\theta_n)) + \sum_n \{\frac{\gamma_n}{2Q_1} . \cos(2\phi-2Q_1(\theta-\theta_n)) . \cos(n(\theta-\theta_n))\}.$$

The function U_2 has at least the same periodicity as f_2. As a first approximation we assume:

$$\frac{\partial U_2}{\partial \theta} = \alpha \cos n\theta + \beta \cos(2\phi-2Q_1\theta) + \kappa \cos(2\phi-2Q_1\theta+n\theta) + \rho \cos(2\phi-2Q_1\theta-n\theta). \tag{20}$$

The coefficients are determined by the requirement (15). The expression for the betatronnumber Q_x now becomes[8]:

$$Q_x = Q_1 - \frac{\Delta}{2} - \frac{\Delta^2}{4(2Q_1-\Delta)} + \sum_n \frac{\gamma_n^2}{2(2Q_1-\Delta)(n^2-(2Q_1-\Delta)^2)} \tag{21}$$

This expression is correct up to the second degree of the field components (thus γ_n^2).

In order to find an expression for the tune which contains higher degree terms in γ_n, we set generally:

$$U_2 = \sum_{k \geq 1} (a_{2k} \cos 2k\phi + b_{2k} \sin 2k\phi). \tag{22}$$

The coefficients a_{2k} and b_{2k} have the same periodicity as the guide field and contain now power series of the field γ_n. Substituting (22) in (15) leads to recurrent relations between these coefficients, from which we can deduce an exact expression for the tune[8]:

$$Q_x = Q_1 - \frac{\Delta}{2} - \frac{\Delta}{2} . \langle b_2 \rangle + \frac{\langle f.b_2 \rangle}{2Q_1} . \tag{23}$$

We have applied this principle on the lattice of the proposed storage ring PAMPUS (see Appendix). In the final expression (23) we considered only the higher order contributions (γ_n^4, γ_n^6 etc.) to the term $\langle f.b_2 \rangle$ those which arise from the field components with the main periodicity (PAMPUS: N=8), i.e. no 'mixed' terms are taken into account. To overcome this, the solving of the recurrent relations becomes undesirably complicated. In table 1 the results of these analytical calculations are compared with numerical matrix calculations with the computer code AGS[9] for several values of the tunes ($Q_x \approx Q_z$). The results are almost independent of the estimation Q_1. The discrepancies for higher values of the tunes are due to the neglected 'mixed' terms. Instead of using an estimation Q_1, which can be based on some foreknowledge, it is possible to choose $\Delta = 0$ in (19). This may lead to a rather wrong estimation Q_1. So one may expect that this more simple method is less accurate especially for higher values of the tunes where the modulation of the field is high. The vertical tune can be calculated analogously[8].

Q_x (AGS)	Q_x (23)	Q_x (21)	Q_x (Δ=0)
1.234	1.235	1.240	1.240
1.498	1.510	1.548	1.553
1.993	1.984	2.110	2.185
2.250	2.197	2.364	2.494
2.549	2.433	2.632	2.837
2.750	2.576	2.790	3.043
3.050	2.754	2.985	3.307

Table 1: Radial betatronnumbers for PAMPUS ($Q_x \approx Q_z$).

Natural chromaticity of a lattice.

The change in the betatronnumber Q with a relative momentum deviation is called chromaticity.

The Hamiltonian which describes the linear motion of a particle with a relative momentum deviation δ is:

$$H^{(2)} = H_u^{(2)} + H_p^{(2)}$$

$$\text{with } H_p^{(2)} = \delta \{\tfrac{1}{2}\varepsilon x_1 \pi_x^2 + \varepsilon p_1 \pi_x \xi - \tfrac{1}{2} \frac{R^2}{\rho^2}(\varepsilon-n)\xi^2\} + \delta \{\tfrac{1}{2}\varepsilon x_1 p_z^2 - \tfrac{1}{2}\frac{R^2}{\rho^2} n z^2 \} . \tag{24}$$

Here we restrict ourselves to homogeneous bending magnets (ε n=0). Sextupoles are also neglected in this section (S=0), so we look for the *natural* chromaticity of a lattice. Our treatment here regards the radial motion. The vertical motion can be handled analogously. Applying now the introduced action and angle variables on the radial part of the Hamiltonian (24) and substituting the Fourier series of the guide field, we get an expression for the natural radial chromaticity[8] :

$$\frac{\Delta Q_x}{\delta} = \{Q.\frac{M_o}{4} - \tfrac{1}{2}\frac{\omega^2}{Q}\}\{1+ \sum_n \frac{(A_n^2+B_n^2)(n^2+(2Q-\Delta)^2)}{(2Q-\Delta)^2(n^2-(2Q-\Delta)^2)^2}\} - \frac{\Delta\,\omega^2}{2(2Q-\Delta)^2}(1+\tfrac{1}{2}M_o) - \frac{\Delta Q}{4(2Q-\Delta)}\{\tfrac{1}{2}M_o + \frac{\omega^2}{Q^2}\} + \sum_n \frac{(A_n^2+B_n^2)}{(2Q-\Delta)^2(n^2-(2Q-\Delta)^2)}\{\tfrac{1}{2}M_o\frac{\omega^2}{Q} - Q\} + - \sum_n \frac{\omega^2(A_n M_n + B_n N_n) + n(A_n R_n - B_n P_n)}{(2Q-\Delta)(n^2-(2Q-\Delta)^2)} \tag{25}$$

$$\text{where } \varepsilon x_1 = \tfrac{1}{2} M_o + \sum_n (M_n \cos n\theta + N_n \sin n\theta) \quad \text{and } \varepsilon p_1 = \tfrac{1}{2} P_o + \sum_n (P_n \cos n\theta + R_n \sin n\theta) \tag{26}$$

are the (reduced) off-energy function and its derivative in the bending magnets (see (9)).

The expression (25) for the natural chromaticity is correct up to the second degree in the Fourier components of the field. This means that we may not expect accurate results for high values of the tunes, analogous to the results of the calculations of the tunes.

We applied the chromaticity equation (25) on the PAMPUS lattice and compare in table 2 the analytical calculations with the results following the normally used theories[4,5]. These theories start from the lattice functions to be known from computer calculations. In these 'classical' theories it is not always quite clear whether the canonical transformations are fully correct[8]. However it must be said that their final expressions have a more simple configuration.

Table 2 presents for the radial oscillation the natural chromaticities for the PAMPUS lattice as calculated by the different methods. Also indicated are the results as given by the orbitprogram AGS[9] which determines the closed orbit of a particle with a relative momentum deviation (here δ=5.10^{-3}) in an iterative way. For lower values of the tunes there is a good agreement between the results of our 'action and angle' method and the 'classical' method[5]. We do not yet understand the discrepancy with the AGS calculations at low values of Q_x.

$Q_x = Q_z$	$\Delta Q_x/\delta$			
	(AGS)	([5]) 'classical'	$(\gamma_t^2\Delta\neq 0)$	$(\gamma_t^2\Delta=0)$
1.50	-0.88	-0.31	-0.36	-0.31
2.00	-1.16	-0.91	-1.12	-0.99
2.25	-1.52	-1.31	-1.53	-1.39
2.55	-2.16	-1.91	-1.99	-1.88
2.75	-2.66	-2.42	-2.25	-2.18
3.05	-3.82	-3.52	-2.57	-2.57

Table 2: Natural radial chromaticities for PAMPUS.

Chromaticity correction by sextupoles.

In order to overcome the head-tail instability in an electron storage ring, one must compensate the negative natural chromaticities by sextupoles. Analogously to the treatment as given previously, we derive an expression for the extra contribution of these sextupoles to the chromaticity. Starting from the Hamiltonian:

$$H = \tfrac{1}{2}\pi_x^2 + \tfrac{1}{2}\frac{R^2}{\rho^2}(\varepsilon-n)\xi^2 - \tfrac{1}{2}\frac{R^2}{\rho^2}\delta x_1 S\,\xi^2 + \tfrac{1}{2}p_z^2 + \tfrac{1}{2}\frac{R^2}{\rho^2}nz^2 + \tfrac{1}{2}\frac{R^2}{\rho^2}\delta x_1 S z^2. \tag{27}$$

One gets for the radial chromaticity, up to the second degree in the Fourier components, due to sextupoles :

$$\left(\frac{\Delta Q_x}{\delta}\right)_{sext} = -\tfrac{1}{4}\frac{S_o}{Q} - \frac{\Delta S_o(4Q-\Delta)}{8Q(2Q-\Delta)^2} + \tag{28}$$
$$- \sum_n \frac{S_o(A_n^2+B_n^2)(n^2+(2Q-\Delta)^2)}{4Q(2Q-\Delta)^2(n^2-(2Q-\Delta)^2)^2} +$$
$$+ \sum_n \frac{S_o(2Q-\Delta)(A_n^2+B_n^2) - 2Q(A_nS_n+B_nT_n)}{2Q(2Q-\Delta)^2(n^2-(2Q-\Delta)^2)}$$

with $\frac{R^2}{\rho^2}x_1 S = \tfrac{1}{2}S_o + \sum_n (S_n \cos n\theta + T_n \sin n\theta)$. (29)

the Fourier expansion of the 'off-energy' function in the sextupole magnets.

It is rather trivial to remark that this equation shows us that the sextupoles are most effective if they are placed at azimuthal positions with extreme values of the off-energy function. Within the expected region of the tunes, application of this equation on PAMPUS agrees with the 'classical' methods[8].

Stability region due to sextupoles.

The non-linear terms of the sextupole fields introduce third order resonances reducing the stable region. We will illustrate our analytical theory on the $Q_x=N/3$ resonance.

The radial Hamiltonian describing the influence of the sextupoles on the stability region is:

$$H = H_u^{(2)} + H_u^{(3)} = H_u^{(2)} - \frac{1}{6}\frac{R^2}{\rho^2}S\,\xi^3. \tag{30}$$

Since we are interested now in the N/3 resonance we apply the following action and angle transformation:

$$\pi_x = \sqrt{2\omega I}.\sin(\phi-N\theta/3) \text{ and } \xi = \sqrt{2I/\omega}.\cos(\phi-N\theta/3). \tag{31}$$

Then the Hamiltonian can be written as:

$$H = e_2.I + f_2.I + e_3.I^{3/2} + f_3.I^{3/2}. \tag{32}$$

In order to skip the oscillating parts we apply the transformation generated by[2]:

$$G(\bar{I},\phi) = -\bar{I}.\phi - \bar{I}.U_2(\theta,\phi) - \bar{I}^{3/2}.U_3(\theta,\phi). \tag{33}$$

In the resulting Hamiltonian we are interested in the last term of this equation. Using the correct functions U_2 and U_3 one finally gets[8]:

$$K = (Q-N/3).\bar{I}+\tfrac{1}{4}\sqrt{2}.\omega^{-3/2}.\{V_n.\cos(3\phi)+W_n.\sin(3\phi)\}.\bar{I}^{3/2}$$

with $-\frac{1}{6}\left(\frac{R^2}{\rho^2}\right)S = \tfrac{1}{2}V_0 + \sum_n (V_n \cos n\theta + W_n \sin n\theta)$. (34)

and ω^2 given by (18).

Since we are here not interested in the tune-shift due to the sextupoles, we only consider the Hamiltonian up to $\bar{I}^{3/2}$. From this Hamiltonian we find the stationary points in the phase-space resulting in the well-known triangle-shaped stability region. Substituting the beam emittance (which is related to $\bar{I}$) we will find the required distance to the resonance line $Q_x = N/3$. Applying this for PAMPUS leads to the same results[8] as given by other methods[10].

Final remarks.

We have shown that our theory starting from direct field quantities can provide all information about the lattice behaviour. Since we started from the basic transformations we can extend this analytical theory to e.g. octupoles, long straight sections and acceleration.

Appendix. PAMPUS lattice.

The lattice of the proposed Dutch dedicated synchrotron radiation source PAMPUS[6] (Photons for Atomic and Molecular Physics and Universal Studies) consists out eight FODO unit cells. One of these cells is sketched with its dimensions in figure 1. PAMPUS will provide at final energy (E = 1.5 GeV) a radiation spectrum with critical wavelength λ_c = 6.9 Å (B_{max} = 1.2 T).

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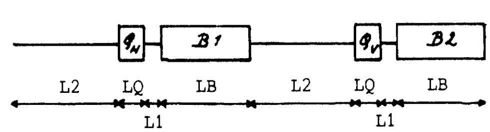
PAMPUS

Photons for Atomic and Molecular Processes and Universal Studies



L1 = 0.300 m LQ = 0.500 m R = 11.450 m
L2 = 2.059 m LB = 1.637 m ρ = 4.170 m

Figure 1. FODO unit cell dimensions for PAMPUS