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A METHOD OF TRANSFER MAPS FOR LINEAR AND NONLINEAR BEAM ELEMENTS*

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Abstract

Transfer matrix methods are widely used to calculate properties of particle orbits as they pass through linear beam elements such as drift spaces, bending magnets, and quadrupoles. A new method of "transfer maps" has been developed to also include nonlinear transformations that result from nonlinear beam elements such as sextupoles, octupoles, etc. The method of transfer maps therefore provides a complete theory of beam transport through both linear and nonlinear elements. In particular, it is possible to use transfer maps in the context of circular machines to study tune shifts, structure resonances, stop band widths, emittance growth rates, etc. Consequently, the method of transfer maps provides an alternative to the method of Hamiltonian perturbation theory usually employed for this purpose.

Introduction

Consider a particle orbit that traverses a given beam element. Let the symbol z^i denote collectively the initial conditions for the particle orbit as it enters the element. For example, z^i may refer to transverse coordinates and momenta or may also be enlarged to include the total momentum and differential transit time. Similarly, let z^f denote the final conditions as the orbit leaves the beam element. Then the connection between the initial and final conditions will be written as

$$z^f = Mz^i, \tag{1}$$

and the relationship M will be called a transfer map for the element in question.

Next, consider a particle orbit that passes through a succession of beam elements. It is clear that there is also a net transfer map for any sequence of beam elements and that this net transfer map is given by the product of the transfer maps of the elements comprising the sequence. In particular, for a circular machine there is a transfer map that describes the effect of making one complete circuit.

Suppose that the coordinate systems used to describe z^i and z^f are selected in such a way that $z^i = z^f = 0$ for a particular reference orbit. This is always possible if the two coordinate systems are independent but requires that the reference orbit be closed if the coordinate systems coincide as would be the case for the transfer map describing one complete circuit in a circular machine. This later possibility offers no particular restriction, however, because it can be shown that circular machines possess a closed orbit under very general conditions.

Now consider orbits near the reference orbit. For these orbits, z^i and z^f will be small, and the connection (1) will have a Taylor expansion of the form

$$z_a^f = \sum R_{ab} z_b^i + \sum T_{abc} z_b^i z_c^i + \ldots \tag{2}$$

Here R will be recognized as the transfer matrix of the usual linear theory and T is the second-order transfer matrix as computed, for example, in the widely used program TRANSPORT.[1] In addition, as indicated, there are generally also third and higher order terms in the expansion.

Because a transfer map arises from tracing orbits whose equations of motion are derived from a Hamiltonian, it must satisfy certain general conditions. Define a matrix L called the linear part of M by the equation.

$$L_{ab}(z^i) = \partial z_a^f / \partial z_b^i. \tag{3}$$

Then a necessary and sufficient condition that M be the result of a Hamiltonian flow is that the matrix L be symplectic.[2] That is, L must satisfy the equation

$$\widetilde{L} J L = J \tag{4}$$

where, in a suitable coordinate system, J is the matrix having the block form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{5}$$

For this reason, a transfer map is technically referred to as a symplectic map.

Note that L in general depends on z^i and that (4) must be satisfied identically for every z^i. It follows that the coefficients R, T, etc., appearing in the expansion (2) are not all independent. Consequently, a Taylor series expansion is not a convenient way of parameterizing a symplectic map.

This paper shows that symplectic maps are more conveniently expressed using the recently developed tools of Lie operators, Lie transformations, and the factorization theorem.[2] These tools also provide a procedure for systematically computing the net transfer map produced by a sequence of linear and nonlinear elements. Finally, they provide a procedure for the multiple iteration of a transfer map, and therefore make it possible to compute the effect of many turns in a synchrotron or storage ring.

General Theory

To introduce the notion of a Lie operator, let f(z) be any function of the phase-space variables z. With each such function there is an associated Lie operator F. This operator acts on functions and is defined by the rule.

$$Fg = [f,g] \quad . \tag{6}$$

Here g is any function of the phase-space variables, and the square bracket [,] denotes the Poisson bracket operation familiar from classical mechanics.

Next, consider the object exp(F), called a Lie transformation, defined by the exponential series

$$\exp(F) = I + F + F^2/2! + F^3/3! + \ldots \tag{7}$$

More explicitly, the action of exp(F) on an arbitrary function g is given by the expression

$$\exp(F)g = g + [f,g] + [f,[f,g]]/2! + \ldots \tag{8}$$

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Using these tools, it can be shown that the general symplectic map of the form (2) can be uniquely factored into a product (usually infinite) of Lie transformations:

$$M = \exp(F_2)\ \exp(F_3)\ \exp(F_4)\ \ldots \qquad (9)$$

Moreover, the Lie operators F_2, F_3 ... are associated with functions f_2, f_3, ... where each f_n is a homogeneous polynomial of degree n in the variables z^i. Thus, for example, in the simplest case of a single coordinate and momentum, f_2 would be a linear combination of the monomials p^2, pq, and q^2.

The factored product representation has several important features. First, the various polynomials have a direct physical interpretation. For example, f_2, or equivalently $\exp(F_2)$, produces the linear transfer matrix part of (2). Similarly, f_3 arises from cubic terms in the equations of motion such as those describing the effect of sextupoles, etc. Second, the factored product can be truncated at any stage and the resulting map will still satisfy the symplectic conditions (3) through (5) exactly. This observation is important because it justifies a particle tracking technique commonly used to simulate the effect of many turns in circular machines.[3] In this technique, nonlinear terms in the Hamiltonian are simulated by delta function impulses that change transverse momenta but leave coordinates unaffected. It can be shown that this technique does indeed produce a symplectic transfer map and that the map thus obtained has a factored product representation whose factors converge rapidly to the true factors of the transfer map for the actual machine as the number of delta function terms is increased. Third, any set of homogeneous polynomials f_n when employed in (9) will produce a symplectic map. This makes it easy to count the number of independent parameters required for a given order of approximation. For example, if TRANSPORT were extended to cover problems through third order at fixed total momentum, then a total of 65 independent parameters would be required to handle the completely general case. (For four variables, there are 10 monomials of degree two, 20 of degree three, and 35 of degree four.) If chromatic effects through third order were also included, which in the static case requires five variables, then at most 120 parameters would be required. Indeed, the use of transfer map methods may be the easiest way to extend TRANSPORT to higher orders.

A Simple Example

A simple example will illustrate the procedure for combining and iterating transfer maps. Consider one-dimensional motion in a weak focusing machine consisting of a perfect ring with one short sextupole insertion. Then the transfer map for going around the ring up to the sextupole is given by $\exp(F_2)$ and the transfer map for the sextupole element is given by $\exp(F_3)$. In this simple case f_2 and f_3 are the homogeneous polynomials

$$f_2 = -\,w(p^2 + q^2)/2 \qquad (10)$$

$$f_3 = sq^3/3\ . \qquad (11)$$

Here $w/(2\pi)$ is the tune of the ring and s measures the path integrated sextupole strength. To verify, for example, that $\exp(F_3)$ really does describe the effect of a short sextupole, it is easily computed using (8) that

$$\exp(F_3)q = q \qquad (12a)$$

$$\exp(F_3)p = p + sq^2\ . \qquad (12b)$$

Combining the perfect ring and the short sextupole, the net transfer map for the entire ring is given by the expression

$$M = \exp(F_2)\ \exp(F_3)\ . \qquad (13)$$

Again using (8) and employing the notation of (1), one finds from (13) that the net transformation in going once around the ring is given by the expressions

$$q^f = q^i \cos w + p^i \sin w \qquad (14a)$$

$$p^f = -q^i \sin w + p^i \cos w + s(q^f)^2. \qquad (14b)$$

This example is essentially the same as that considered by Crosbie et al.[4] However, for reasons of symmetry, they chose to view the entire ring as half a sextupole followed by the perfect ring followed by the second half of the sextupole. In the language of Lie transformations, their choice amounts to a transfer map M_C having the factorization $\exp(F_3/2)\ \exp(F_2)\ \exp(F_3/2)$. The analog to (14) in this case is an expression containing cubic and quartic terms as well as quadratic terms. It is also interesting to note that Henon[5] has studied a map M_H in an astronomical context that proves to have the factorization $\exp(F_3)\ \exp(F_2)$. (Note that the order in which Lie operators and Lie transformations are written is important because Lie operators do not in general commute!) Routine computation shows that both these maps are related to M and to each other by simple nonlinear "similarity" transformations,

$$M_C = \exp(F_3/2)\ M\ \exp(-F_3/2) \qquad (15)$$

$$M_H = \exp(F_3)\ M\ \exp(-F_3)\ . \qquad (16)$$

Thus, problems that initially appear to be very different are in fact essentially the same. The relative ease with which such unsuspected relationships can be discovered and stated is an illustration of the power of the method of transfer maps.

Suppose one now wants to compute the effect of going around the ring many times. This is equivalent to computing M^n for large n. The computation of M^n would be easy if a Lie operator H could be found such that M could be re-expressed in the form $\exp(H)$, for then M^n would be simply given by $\exp(nH)$. The determination of such an H is a standard problem in the theory of Lie algebras that is solved by using the Campbell-Baker-Hausdorff formula.[2] This formula gives H in terms of F_2, F_3, etc., and their multiple commutators. In addition, there is an analogous formula that gives the function h associated with H in terms of f_2, f_3, etc., and their multiple Poisson brackets. Finally, it can be shown that the computation of $\exp(nH)$ is equivalent to the integration of a "trajectory" in "z space" for n units of "time" using -h as an "effective" Hamiltonian.

For the example under consideration, h is given by the formal operator formula,

$$h = f_2 + F_2\left[1 - \exp(-F_2)\right]^{-1} f_3 + \ldots \qquad (17)$$

and more explicitly by the expansion

$$-h = r^2/2 + (sr^3/8)\ \left[(\cos 3\psi)/\sin(3w/2) + (\cos \psi)/\sin(w/2)\right] + O(s^2r^4)\ . \qquad (18)$$

Here r,ψ are certain polar variables in the q,p plane.

As is well known, a machine consisting of a perfect ring and a sextupole exhibits nonlinear structure resonances whenever the tune has one-third integer values. The Campbell-Baker-Hausdorff formula acknowledges this fact by diverging at these tune values. Note that the term sin (3w/2) appearing in (18) vanishes at these values. However, the map M^3 can still be written in simple exponential form at and near resonance values and has a well-behaved Campbell-Baker-Hausdorff formula. Therefore, resonance behavior can be easily explored. Writing M^3 as $\exp(3H_r)$, one finds, for example, near the resonance $w = 2\pi/3$ that

$$h_r = (w - 2\pi/3)h + O(s^2r^4) \quad . \tag{19}$$

A Numerical Comparison

The points in the figures below show the result of iterating the transfer map M numerous times for various initial conditions and tunes. For comparison, the circles show the result of integrating "trajectories" using the appropriate effective Hamiltonian. Iterates of M are compared with "trajectories" derived from h in the non-resonant case and iterates of M^3 are compared with "trajectories" derived from h_r in the near resonant case. The agreement between exact results and those given by the effective Hamiltonians is good. If agreement were perfect, the circles would be centered on the dots. Agreement could be improved further by carrying out the Campbell-Baker-Hausdorff formula to higher order in s. In both figures, q and p range from -1 to 1.

In Fig. 1 the tune is away from resonance, the sextupole produces only an "egg shaped" distortion, and its effect appears to be perfectly described by h. Since the trajectories generated by h are closed, all initial conditions within the square give stable orbits.

In Fig. 2 the tune is near a third integer. In this case there is some loss in "phase accuracy", i.e., the circles are no longer centered on the points after a large number of turns. However, the agreement in "shape" is still good. Observe that trajectories corrresponding to large betatron amplitudes go off to infinity. Consequently, h_r correctly predicts that motion near resonance becomes unstable if the betatron amplitude is too large.

Examination of the figures shows that the effective Hamiltonians reproduce the general overall features of the transfer map at all tune values. Consequently, the effective Hamiltonians generalize the Courant-Snyder invariants, and moreover can be used to determine regions of stability and instability, the dependence of tune on betatron amplitude, beating ranges, resonance widths, and rates of emittance growth.

Conclusions

The method of transfer maps reproduces in an efficient and unified manner all results that have been previously obtained from current beam transport and Hamiltonian perturbation theories. It appears to offer the promise of going beyond these theories in some areas where the current theories become too complicated for easy application.

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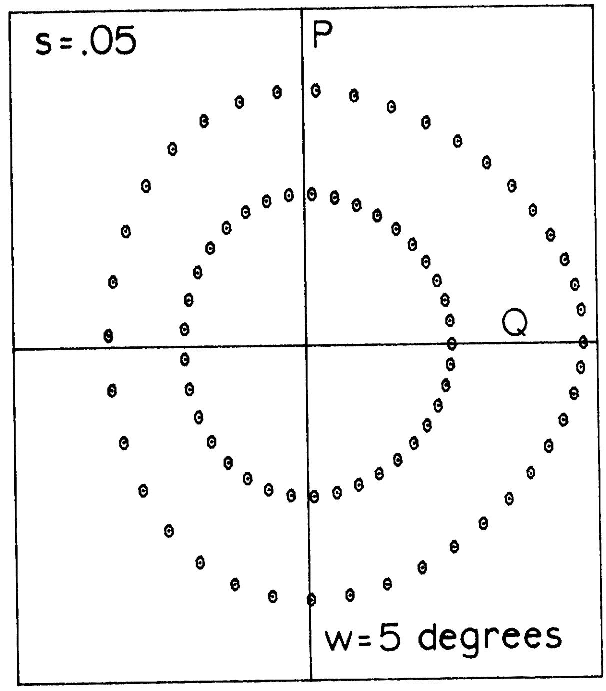

Fig. 1. Typical behavior when the tune is far from resonance. Orbits are stable, and iterates of M appear to agree perfectly with trajectories derived from h.

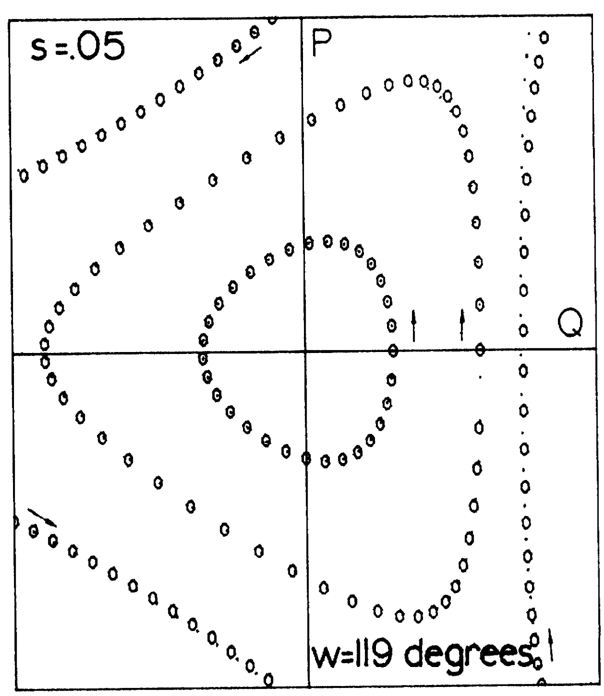

Fig. 2. Behavior near a third integer resonance. Motion is unstable for large betatron amplitudes. All of phase space becomes unstable exactly at resonance. Evidently the location of the stable region and the growth in the unstable region are well described by h_r.