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STABILITY OF A LARGE SYNCHROTRON WITH CLOSED ORBIT DISTORTIONS

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Summary and Introduction

In a large synchrotron or a storage ring, the random closed orbit distortions are generally large. When combined with the sextupole magnets for chromaticity correction, they produce linear stopbands because effective gradient errors appear. Further, the vertical closed orbit distortion produces a vertical dispersion because there appears a vertical bend in the quadrupole and sextupole magnets. These effects are studied by an analytical method.

It is shown that the tune ν, which is close to $\nu = mNs$ and $3\nu = mNs$ (where Ns is the number of superperiods and m is an arbitrary integer) should be avoided to have a small stopband width. In order to have a small vertical dispersion, the tune close to $2\nu = mNs$ should be avoided. The present theory also indicates a preferable method of chromaticity correction to avoid an excessive vertical dispersion. The effect of closed orbit correction is not considered in this paper.

Equation of Motion

The Hamiltonian of particle motion in a machine with sextupole magnets, a momentum error $\Delta p/p$ and horizontal and vertical kick errors θ_{xi} and θ_{yi} is given as[1,2]

$$H = H_0 + V$$

$$H_0 = \frac{1}{2}(p_x^2+p_y^2) + \frac{1}{2}K(x^2-y^2)$$

$$V = \sum_{i=1}^{M}\theta_{xi}\delta(z-z_i)(1-\frac{\Delta p}{p})x - \frac{1}{\rho}\frac{\Delta p}{p}x - \sum_{i=1}^{M}\theta_{yi}\delta(z-z_i)(1-\frac{\Delta p}{p})y - \frac{1}{2}K\frac{\Delta p}{p}(x^2-y^2) + \frac{1}{6}K'(1-\frac{\Delta p}{p})(x^3-3xy^2), \quad (1)$$

where

$$K = \frac{1}{B\rho}\frac{\partial B_y}{\partial x}$$

$$K' = \frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2}$$

x = horizontal deviation

y = vertical deviation

z = orbit length (independent variable)

p_x, p_y = canonical momentum

z_i = orbit position of a kick

M = number of kick elements

The kicks θ_{xi} and θ_{yi} are given by

$$\theta_{xi} = \frac{\Delta B_{yi}\ell_i}{B\rho}, \quad \theta_{yi} = \frac{\Delta B_{xi}\ell_i}{B\rho}, \quad (2)$$

where $\Delta B_{x,y}$ denotes a random field error and ℓ_i is the length of the kick element. We assumed a thin lens (δ-function) for a kick.

The closed orbits (including dispersion functions) x_{eq} and y_{eq} are periodic solutions of the Hamiltonian equations and thus are described by the equations

$$x''_{eq} + Kx_{eq} = \frac{1}{\rho}\frac{\Delta p}{p} - \sum_{i=1}^{M}\theta_{xi}\delta(z-z_i)(1-\frac{\Delta p}{p}) + K\frac{\Delta p}{p}x_{eq} - \frac{1}{2}K'(1-\frac{\Delta p}{p})(x_{eq}^2-y_{eq}^2), \quad (3)$$

$$y''_{eq} - Ky_{eq} = \sum_{i=1}^{M}\theta_{yi}\delta(z-z_i)(1-\frac{\Delta p}{p}) - K\frac{\Delta p}{p}y_{eq} + K'(1-\frac{\Delta p}{p})x_{eq}y_{eq}, \quad (4)$$

where primes on x_{eq} and y_{eq} denote differentiation with respect to z.

Further simplification can be made by putting

$$x_{eq} = x_{co} + \eta_x\frac{\Delta p}{p},$$
$$y_{eq} = y_{co} + \eta_y\frac{\Delta p}{p}, \quad (5)$$

where "co" denotes random closed orbit distortions and η_x and η_y are horizontal and vertical dispersions. Keeping terms of zeroth and first orders in $\Delta p/p$, neglecting nonlinear terms and assuming that $\eta_y \ll \eta_x$ and η_x is given by the $1/\rho$ term in eq.(3), we obtain

$$y''_{co} - Ky_{co} = \sum_{i=1}^{M}\theta_{yi}\delta(z-z_i) \quad (6)$$

$$\eta''_y - K\eta_y = \sum_{i=1}^{M}\theta_{yi}\delta(z-z_i) - (K-K'\eta_x)y_{co}. \quad (7)$$

Eq.(6) is well-known[3] and eq.(7) has been derived by several authors.[4,5]

We then consider a betatron oscillation around the closed orbits x_{eq} and y_{eq}. By putting

$$x = x_{eq} + u, \quad y = y_{eq} + v, \quad (8)$$

where u and v denote the amplitudes of betatron oscillation, the perturbation Hamiltonian V is transformed to[1]

$$V = \frac{1}{2}\frac{\Delta p}{p}K(u^2-v^2) - \frac{1}{2}K'x_{eq}(1-\frac{\Delta p}{p})(u^2-v^2) + K'y_{eq}(1-\frac{\Delta p}{p})uv, \quad (9)$$

where we have neglected nonlinear terms since we consider only linear stopbands.

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Vertical Dispersion

Solutions of eqs.(6) and (7) are given in terms of Fourier series[2]

$$y_{co} = \sqrt{\beta_y} \sum_{k=-\infty}^{\infty} \frac{\nu_y^2 f_k}{\nu_y^2 - k^2} e^{ik\phi_y} \tag{10}$$

with

$$f_k = \frac{1}{2\pi\nu_y} \sum_{i=1}^{M} \beta_{yi}^{1/2} \theta_{yi} e^{-ik\phi_{yi}} \tag{11}$$

and

$$\eta_y = \sqrt{\beta_y} \sum_{n=-\infty}^{\infty} \frac{\nu_y^2 f_n e^{in\phi_y}}{\nu_y^2 - n^2}$$

$$- \frac{\sqrt{\beta_y}}{2\pi} \sum_{k,n=-\infty}^{\infty} \frac{\nu_y^3 f_k J_{n-k}}{(\nu_y^2-k^2)(\nu_y^2-n^2)} e^{in\phi_y} \tag{12}$$

with

$$J_m = \int_0^C \beta_y (K-K'\eta_x) e^{-im\phi_y} dz. \tag{13}$$

Notations are familiar ones of Courant and Snyder.[3] The first term in eq.(12) is equal to y_{co} and small. The function J_m is related to $\Delta\beta/\beta$ and chromaticity in the way

$$\frac{(\Delta\beta/\beta)_y}{\Delta p/p} = \frac{\nu_y}{4\pi} \sum_{m=-\infty}^{\infty} \frac{J_m e^{im\phi}}{\nu_y^2-(m/2)^2} \tag{14}$$

$$\frac{\Delta\nu_y}{\Delta p/p} = \frac{1}{4\pi} J_0. \tag{15}$$

Looking at eq.(12), we see that the largest contribution to η_y comes from the terms $n \sim \pm\nu_y$ and $k \sim \pm\nu_y$. Since $n-k$ should be an integer multiple of superperiodicity N_s, we see that the tune around $2\nu_y \sim N_s n$ (n = arbitrary integer) should be avoided to have a small vertical dispersion. Even then, the largest contribution to η_y comes from J_0 and J_m where $m \sim \pm 2\nu$. Thus, it is important in chromaticity correction to make these terms small. It is to be noted here that Close et al.[6] has pointed out by a computational method that the vertical dispersion at the half-integral structure resonance described above is rather easily corrected by closed orbit correction.

Horizontal dispersion is also affected by random closed orbit distortions. The additional horizontal dispersion $\Delta\eta_x$ is given in a similar way by

$$\Delta\eta_x = \sqrt{\beta_x} \sum_{n=-\infty}^{\infty} \frac{\nu_x^2 f_n}{\nu_x^2 - n^2} e^{in\phi_x}$$

$$+ \frac{\sqrt{\beta_x}}{2\pi} \sum_{n,k=-\infty}^{\infty} \frac{\nu_x^3 f_k J_{n-k}^x}{(\nu_x^2-k^2)(\nu_x^2-n^2)} e^{in\phi_x} \tag{16}$$

with

$$f_k = \frac{1}{2\pi\nu_x} \sum_{i=1}^{M} \beta_{xi}^{1/2} \theta_{xi} e^{-ik\phi_{xi}} \tag{17}$$

$$J_m^x = \int_0^C \beta_x (K-K'\eta_x) e^{-im\phi_x} dz. \tag{18}$$

Half-Integral Resonance

As seen from the Hamiltonian (9), the horizontal closed orbit distortion gives rise to half-integral resonance and the effective gradient error is equal to $-K'x_{eq}$. We consider a case for a right momentum or $\Delta p/p = 0$. The stopband width for half-integral resonance is given by[3]

$$\delta\nu_{x,y} = \frac{1}{2\pi} \left| \int_0^C \beta_{x,y} x_{eq} K' \exp(ip\phi_{x,y}) dz \right| \tag{19}$$

for $\nu \sim p/2$ (p: ingeter).

The rms estimate is expressed as

$$(\delta\nu_{x,y})^2_{rms} = \frac{1}{4\pi^2} \int_0^C dz \int_0^C dz' \beta_{x,y}(z)\beta_{x,y}(z')K'(z)K'(z')$$

$$\sqrt{\beta_x(z)\beta_x(z')}\ \overline{H(z)H(z')}\ \mathrm{cosp}(\phi-\phi'), \tag{20}$$

where we put

$$x_{eq} = \sqrt{\beta_x}\, H. \tag{21}$$

To proceed further, we should evaluate the function $\overline{H(z)H(z')}$.[7] If we assume a thin lens for a kick θ_i, $H(\phi)$ is expressed as

$$H(\phi) = \frac{1}{2\sin\pi\nu} \left[\sum_{i=1}^{M} \beta_i^{1/2}\ \theta_i \cos\nu(\pi+\phi-\psi_i) \right.$$

$$\left. + 2\sin\pi\nu \sum_{i=1}^{M(\phi)} \beta_i^{1/2}\ \theta_i \sin\nu(\phi-\psi_i) \right], \tag{22}$$

where M is the total number of kick elements and $M(\phi)$ is the number of kick elements located at the azimuthal points from 0 to ϕ.

We now assume that the kicks θ_i's are "uncorrelated", i.e. $\overline{\theta_i\theta_j} = \overline{\theta_i^2}\,\delta_{ij}$, where the bar indicates an ensemble average. This assumption is reasonable when we consider random closed orbit distortions. When an orbit correction is taken into account, the corrector kick will be correlated in some way to the random kicks. We neglect this case in this paper. Then,

$$\overline{H(\phi)H(\phi')} = \frac{1}{4\sin^2\pi\nu} \left[\sum_{i=1}^{M} \beta_i \overline{\theta_i^2} \cos\nu(\pi+\phi-\psi_i)\cos\nu(\pi+\phi'-\psi_i) \right.$$

$$+ 2\sin\pi\nu \sum_{i=1}^{M(\phi)} \beta_i \overline{\theta_i^2} \sin\nu(\phi-\psi_i)\cos\nu(\pi+\phi'-\psi_i)$$

$$+ 2\sin\pi\nu \sum_{i=1}^{M(\phi')} \beta_i \overline{\theta_i^2} \sin\nu(\phi'-\psi_i)\cos(\pi+\phi-\psi_i) \tag{23}$$

$$\left. + 4\sin^2\pi\nu \sum_{i=1}^{M(\phi,\phi')} \beta_i \overline{\theta_i^2} \sin\nu(\phi-\psi_i)\sin\nu(\phi'-\psi_i) \right],$$

where $M(\phi,\phi')$ denotes the minimum of $M(\phi)$ and $M(\phi')$.

Another form of $\overline{H(\phi)H(\phi')}$ is obtained from the Fourier series form of closed orbit distortion given by eqs.(10) and (11). In this case,

$$\overline{H(\phi)H(\phi')} = \sum_{k,k'=-\infty}^{\infty} \frac{\nu^4\ \overline{f_k f_{k'}}}{(\nu^2-k^2)(\nu^2-k'^2)} e^{ik\phi+ik'\phi'} \tag{24}$$

The ensemble mean $\overline{f_k f_{k'}}$ is given for "uncorrelated" thin lens kicks in the form

$$\overline{f_k f_{k'}} = \frac{1}{4\pi^2\nu^2}\sum_{i=1}^{M}\beta_i\overline{\theta_i^2}\, e^{-i(k+k')\psi_i} \tag{25}$$

$$= F_{k+k'} .$$

If $\overline{\theta_i^2}$ is equal to $\overline{\theta^2}$ for all kicks, the function is non-zero only when $k+k' = mN_s$ (m: arbitrary integer). We consider this case.

If we insert eq.(23) or (24) into eq.(20), we obtain the rms stopband width for half-integral resonance.

Linear Stopbands

We give the formulae[7] of rms linear stopband widths and linear tune shifts. The expressions are given in terms of Fourier series because qualitative features of the present theory are more manifest. For half-integral resonance, the stopband widths are given as ($\nu \sim p/2$)

$$(\delta\nu_x)^2_{rms} = \sum_{m,m'=-\infty}^{\infty} B_m B_{m'} \frac{\nu^4 F_{(m+m')}}{\{\nu^2-(p+m)^2\}\{\nu^2-(p-m')^2\}} , \tag{26}$$

$$(\delta\nu_y)^2_{rms} \approx \sum_{m,m'=-\infty}^{\infty} B'_m B'_{m'} \frac{\nu^4 F_{(m+m')}}{\{\nu^2-(p+m)^2\}\{\nu^2-(p-m')^2\}} . \tag{27}$$

The linear tune shifts $\Delta\nu_x$, $\Delta\nu_y$ are given as

$$(\Delta\nu_x)^2_{rms} = \frac{1}{4}\sum_{m,m'=-\infty}^{\infty} B_m B_{m'} \frac{\nu^4 F_{(m+m')}}{\{\nu^2-m^2\}\{\nu^2-m'^2\}} , \tag{28}$$

$$(\Delta\nu_y)^2_{rms} = \frac{1}{4}\sum_{m,m'=-\infty}^{\infty} B'_m B'_{m'} \frac{\nu^4 F_{(m+m')}}{\{\nu^2-m^2\}\{\nu^2-m'^2\}} , \tag{29}$$

The stopband width for the sum resonance ($\nu_x+\nu_y \sim p$) is given as

$$(\delta\nu_{sum})^2_{rms} \approx \sum_{m,m'=-\infty}^{\infty} B'_m B'_{m'} \frac{\nu^4 F'_{(m+m')}}{\{\nu^2-(p+m)^2\}\{\nu^2-(p-m')^2\}} , \tag{30}$$

The width for difference resonance ($\nu_x-\nu_y \sim 0$) is

$$(\delta\nu_{dif})^2_{rms} \approx \sum_{m,m'=-\infty}^{\infty} B'_m B'_{m'} \frac{\nu^4 F'_{(m+m')}}{\{\nu^2-m^2\}\{\nu^2-m'^2\}} . \tag{31}$$

In the above formulae

$$B_m = \frac{1}{2\pi}\sum_i \beta_{xi}^{3/2} K'_i \ell_i \cos(m\phi_i)$$

$$B'_m = \frac{1}{2\pi}\sum_i \beta_{xi}^{1/2}\, \beta_{yi}\, K'_i \ell_i \cos(m\phi_i) \tag{32}$$

$$F_m = \frac{\overline{\theta^2}}{4\pi^2\nu^2}\sum_i \beta_{xi} \cos(m\psi_{xi})$$

$$F'_m = \frac{\overline{\theta^2}}{4\pi^2\nu^2}\sum_i \beta_{yi} \cos(m\psi_{yi}) .$$

We have assumed that the lattice is symmetric and $\nu_x = \nu_y = \nu$. m,m' in the summation takes values which are integer multiples of Ns. Suffix x or y is attached to ϕ_i depending on the case.

The stopband width for half-integral and sum resonances becomes large when $\nu \sim \pm(m-p)$ and $\nu \sim \pm(m'-p)$. Since $p \sim 2\nu$, this condition is expressed as $\nu = N_s n$, $3\nu = N_s n$ (n: arbitrary integer). On the other hand, the tune shift and the stopband width for difference resonance become large only when $\nu = N_s n$.

Conclusion

It is shown that the tunes, $\nu \sim N_s n$, $2\nu \sim N_s n$ and $3\nu \sim N_s n$, are unfavorable because they give rise to a large dispersion or a large stopband width for linear resonances. These conditions are the same as those for integral, half-integral and third-integral resonances. We have considered linear resonaces for $\Delta p/p = 0$, but we expect a similar resonance effect for $\Delta p/p \neq 0$ due to the presence of η_y and $\Delta\eta_x$.

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