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COUPLING RESONANCE $2\nu_H - 2\nu_V = 0$ IN THE KEK PROTON SYNCHROTRON

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Summary

The width of the fourth-order coupling resonance $2\nu_H - 2\nu_V = 0$ is calculated for the KEK proton synchrotron for the case of various octupolar nonlinear fields. The space charge nonlinear field has the largest effect and it may cause some beam loss for the diagonal tune $\nu_H \sim \nu_V \sim 7.25$ at a high intensity. This coupling resonance is avoided by choosing an off-diagonal tune $\nu_H \sim 7.25$ and $\nu_V \sim 6.25$. Even then, the linear tune shift and the tune spread due to space charge octupolar field are found to be rather large and these may be connected with the beam loss in the KEK proton synchrotron.

Introduction

In the KEK proton synchrotron, it was found experimentally[1] that the off-diagonal tune $\nu_H \sim 7.25$ and $\nu_V \sim 6.25$ showed a better transmission of beams than the diagonal tune $\nu_H \sim \nu_V \sim 7.25$. Here ν_H and ν_V are horizontal and vertical tunes. This suggests the effect of a coupling resonance of the type $n\nu_H - n\nu_V = p$, where n and p are arbitrary integers. Since the KEK proton synchrotron has a superperiodicity of four, the above resonance is avoided when p is not an integer multiple of four, i.e. if we choose the tunes such that $|\nu_H - \nu_V| = 1$.

The linear coupling resonance (n=1) is excited by a skew quadrupolar component, which is absent in a machine designed with a midplane symmetry and is only due to errors. Since the error field does not possess the superperiodicity character, the above argument is not applied. Thus, we are led to a consideration of an octupolar resonance (n=2).

There are several sources of octupolar fields. In the field measurement on the magnets for the KEK proton synchrotron,[2] a small octupole component is reported for the quadrupole magnet, which is due to the lack of exact quadrupolar symmetry. Further, the quadrupole magnet has a dodecapole component. When this dodecapole component is combined with a closed orbit displacement due to momentum spread, it produces an effective octupole field. It is also known[3,4] that the fringing field of a quadrupole magnet produces an octupole component. Space charge force is another source of an octupolar field.[5]

The coupling resonance $2\nu_H - 2\nu_V = 0$ due to these octupole components is studied for a diagonal tune $\nu_H \sim \nu_V \sim 7.25$. The space charge effect is found to be the largest. It is also shown that the linear tune shift and the tune spread due to space charge octupolar field are rather large.

Hamiltonian

The relevant Hamiltonian is expressed as

$$H(x,y,p_x,p_y;\ z) = H_0 + V\ ,$$
$$H_0 = \frac{1}{2}(p_x^2+p_y^2) + \frac{1}{2}K(x^2-y^2), \tag{1}$$
$$K = \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\ ,$$

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where x and y are horizontal and vertical coordinates, p_x and p_y are momenta canonically conjugate to x and y, z is the orbit length taken as an independent variable, and V denotes the perturbation Hamiltonian due to nonlinear fields. The form of V depends on the source of nonlinear fields and we will give it separately in the followings.

i) octupole component of a quadrupole magnet

$$V = \frac{1}{24}K^{(2)}(x^4-6x^2y^2+y^4),\qquad K^{(2)} = \frac{\partial^2 K}{\partial x^2}\ . \tag{2}$$

ii) dodecapole component and closed orbit displacement

The Hamiltonian for the dodecapole component is

$$V = \frac{1}{6!}K^{(4)}(x^6-15x^4y^2+15x^2y^4-y^6),\qquad K^{(4)} = \frac{\partial^4 K}{\partial x^4}\ . \tag{3}$$

We consider a case when the closed orbit is displaced by $x_{eq} = x_p\frac{\Delta p}{p}$ due to the momentum error $\Delta p/p$, where x_p is the dispersion function. Then, if we put $x = x_{eq} + u$, where u is the amplitude of betatron oscillation, the relevant Hamiltonian becomes

$$V = \frac{1}{48}K^{(4)}\ x_p^2\left(\frac{\Delta p}{p}\right)^2(u^4-6u^2y^2+y^4). \tag{4}$$

iii) fringing field of a quadrupole magnet

The scalar potential ψ of a quadrupole magnet with a fringing field is given by[3,4]

$$\psi = \{Kxy - \frac{1}{12}K''xy(x^2+y^2)\}B\rho\ ,\qquad K'' = \frac{d^2K}{dz^2}\ , \tag{5}$$

and the relevant fourth-order Hamiltonian is given by

$$V = \frac{1}{4}K'xy(p_xy-p_yx) - \frac{1}{48}K''(x^4-y^4),\qquad K' = \frac{dK}{dz}\ . \tag{6}$$

The relation between the scalar potential and the Hamiltonian is given in ref.(6).

iv) space charge

The fourth-order Hamiltonian for the space charge force for the Gaussian beam distribution is given by[5]

$$V = \frac{\lambda\, r_p}{B\beta^2\gamma^3}\left[\frac{2a+b}{3a^3(a+b)^2}x^4 + \frac{2}{ab(a+b)^2}x^2y^2 + \frac{2b+a}{3b^3(a+b)^2}y^4\right], \tag{7}$$

where λ is the number density of particles per unit length, r_p is the classical proton radius, B is the bunching factor, β and γ are usual relativistic factors, and a and b are horizontal and vertical beam sizes of $\sqrt{2}$ times the standard deviation.

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v) image force

The image force Hamiltonian is derived by the Taylor expansion of the formula of Laslett,[7] assuming a parallel plate vacuum chamber of half-height h and a parallel plate magnet pole of half-height g and filling factor of ρ/R. The expression is

$$V = \frac{r_p}{\beta^2\gamma}\Big[-\frac{7}{720}\Big(\frac{\pi}{2h}\Big)^4\frac{N}{2\pi RB} + \frac{1}{90}\Big(\frac{\pi}{2g}\Big)^4\frac{N\beta^2}{2\pi R}\frac{\rho}{R} + \frac{7}{720}\Big(\frac{\pi}{2h}\Big)^4\frac{N\beta^2}{2\pi R}\Big(\frac{1}{B}-1\Big)\Big](x^4-6x^2y^2+y^4), \tag{8}$$

where N is the number of protons and R is the mean radius of the machine.

Averaged Hamiltonian

Except for the fringing field, the perturbation Hamiltonian has the form

$$V = A(z)x^4 + B(z)x^2y^2 + C(z)y^4 \ . \tag{9}$$

We first consider this Hamiltonian. We make the transformation of variables from (x, p_x) and (y, p_y) to (I_x, ψ_x) and (I_y, ψ_y), which are given by the relation

$$x,y = \sqrt{2I_{x,y}\beta_{x,y}}\cos(\nu_{H,V}\phi_{x,y} + \psi_{x,y}) \ ,$$

$$p_x,p_y = -\sqrt{\frac{2I_{x,y}}{\beta_{x,y}}}\{\sin(\nu_{H,V}\phi_{x,y} + \psi_{x,y}) + \alpha_{x,y}\cos(\nu_{H,V}\phi_{x,y} + \psi_{x,y})\} \ . \tag{10}$$

Here $\beta_{x,y}$, $\alpha_{x,y}$ and $\phi_{x,y}$ are an amplitude function, its derivative and a phase function of Courant and Snyder.[8] This is a canonical transformation whose generating function is

$$F(x,\psi_x,y,\psi_y;z) = -\frac{\alpha_x}{2\beta_x}x^2 - \frac{x^2}{2\beta_x}\tan(\nu_H\phi_x+\psi_x) - \frac{\alpha_y}{2\beta_y}y^2 - \frac{y^2}{2\beta_y}\tan(\nu_V\phi_y+\psi_y). \tag{11}$$

When this transformation is done, the unperturbed Hamiltonian becomes zero and the perturbed Hamiltonian is given by (9) where the transformation of variables of (10) is made. It is then evident that $I_{x,y}$ and $\psi_{x,y}$ are constants when perturbation is not considered. $2I_{x,y}$ are emittances of the beam and $\psi_{x,y}$ are phase angles. We further transform the independent variable from z to $\theta = z/R$. Then, the Hamiltonian is

$$V = R[4A\beta_x^2I_x^2\cos^4(\nu_H\phi_x+\psi_x) + 4B\beta_x\beta_yI_xI_y\cos^2(\nu_H\phi_x+\psi_x) \times \cos^2(\nu_V\phi_y+\psi_y) + 4C\beta_y^2I_y^2\cos^4(\nu_V\phi_y+\psi_y)]. \tag{12}$$

We then average this Hamiltonian and keep the terms which vary slowly under the condition $2\nu_H - 2\nu_V = p + 2\delta$ $(\delta \ll 1)$.[9] Then, the averaged Hamiltonian $\langle V\rangle$ is

$$\langle V\rangle = \frac{3}{2}A_0I_x^2 + \frac{3}{2}C_0I_y^2 + I_xI_y\Big[B_0 + \frac{|B_p|}{2} \times \cos2(\psi_x-\psi_y+\delta\theta+b_p)\Big], \tag{13}$$

where

$$A_0 = \frac{1}{2\pi}\int A\,\beta_x^2\,dz,$$
$$C_0 = \frac{1}{2\pi}\int C\,\beta_y^2\,dz, \tag{14}$$
$$B_0 = \frac{1}{2\pi}\int B\,\beta_x\beta_y\,dz,$$
$$|B_p|e^{ib_p} = \frac{1}{2\pi}\int B\beta_x\beta_y e^{i(2\nu_H\phi_x-2\nu_V\phi_y-2\nu_H\theta+2\nu_V\theta+p\theta)}dz.$$

Practically $|B_p|e^{ib_p}$ is the p-th Fourier harmonic since $\phi_x \sim \phi_y \sim \theta$. As will be shown later, the quantity $|B_p|e^{ib_p}$ is approximately real for the KEK proton synchrotron so that it is replaced by B_p.

Then, we can perform the further transformation from (ψ_x, I_x), (ψ_y, I_y) to (w_x,I_x), (w_y,I_y), where

$$w_x = \psi_x + \delta\theta \text{ and } w_y = \psi_y.$$

Now the Hamiltonian is

$$\langle V\rangle = \delta I_x + \frac{3}{2}A_0I_x^2 + \frac{3}{2}C_0I_y^2 + I_xI_y\Big[B_0+\frac{B_p}{2}\cos2(w_x-w_y)\Big]. \tag{15}$$

It is seen that the Hamiltonian is a constant of motion and also

$$I_x + I_y = J \tag{16}$$

is constant.

The fringing field Hamiltonian of the form

$$V = Dxy\ (p_xy - p_yx) \tag{17}$$

has an additional contribution

$$\langle V\rangle = I_xI_y\ \{D_0 + \frac{1}{2}D_p\cos2(w_x - w_y)\}\ , \tag{18}$$

where

$$D_0 = \frac{1}{2\pi}\int D(\alpha_x\beta_y - \alpha_y\beta_x)dz, \tag{19}$$

$$D_p = \frac{1}{2\pi}\int D\{(i-\alpha_x)\beta_y-(i-\alpha_y)\beta_x\} \times e^{i\{2\nu_H\phi_x-2\nu_V\phi_y-2\nu_H\theta+2\nu_V\theta+p\theta\}}dz\ .$$

Numerical Estimate

The coefficients of the averaged Hamiltonian (15) and (18) are summarized in Table 1. Since B_p is approximately equal to B_0 for the diagonal tune $2\nu_H - 2\nu_V = 0$ or $p = 0$, only the value of B_0 is given. The values are at injection or at 500 MeV.

For this calculation, the results of the field measurement[2] are used. For the dodecapole component effect, the momentum spread of $\Delta p/p = 2.7\times10^{-3}$ is assumed. For the fringing field, a step-funtion fringing field is taken. For the space charge effect, the intensity of 4.5×10^{12} ppp is assumed and we have used a smooth approximation for the beam size.

It is seen that the space charge has by far the largest effect and we consider only the space charge effect in the following.

Table 1
Coefficients of the Averaged Hamiltonian (unit m^{-1})

	A_0	$B_0 \approx B_p$	C_0
octupole component	6.1	0	-6.1
dodecapole component ($\frac{\Delta p}{p} = 2.7\times10^{-3}$)	15.7	-12.3	-2.9
space charge	3.60×10^3	1.18×10^4	9.92×10^3
image force	-15.9	96.3	-15.9

	A_0	C_0	D_0	D_p
fringing field	-15.1	15.1	-34.2	34.2

Dynamics

Following Montague,[5] we transform the variables from (w_x, I_x), (w_y, I_y) to (w_x, J), (ψ,α), where

$$J = I_x + I_y ,$$
$$\psi = 2(w_x - w_y), \tag{20}$$
$$\alpha = I_y/J - \frac{1}{2} .$$

This is a combination of a canonical and a scale transformation and the Hamiltonian is given by

$$\langle V\rangle = \Delta Q_e[(\eta-\cos\psi)\alpha^2+\chi\alpha+\frac{1}{4}\cos\psi] , \tag{21}$$

$$\Delta Q_e = B_0 J ,$$

$$\eta = \frac{3(A_0+C_0)}{B_0} - 2, \tag{22}$$

$$\chi = \frac{-2\delta + 3J(C_0-A_0)}{B_0 J} .$$

Here α and ψ are canonical variables.

The extrema of α is given by $\frac{d\alpha}{d\psi} = 0$, which leads to

$$(\alpha^2 - \frac{1}{4})\sin\psi = 0 . \tag{23}$$

The beating range of α is given by eq.(21) with constant $\langle V\rangle$ and the limit is given by the curves

$$(\eta-1)\alpha^2 + \chi\alpha + \frac{1}{4} = C_+ \text{ (const)},$$
$$(\eta+1)\alpha^2 + \chi\alpha - \frac{1}{4} = c_- \text{ (const)}. \tag{24}$$

The fixed points are given by $\frac{d\alpha}{d\theta} = \frac{d\psi}{d\theta} = 0$, which occur at

$$(\alpha^2 - \frac{1}{4})\sin\psi = 0 ,$$

$$\alpha = - \frac{\chi}{2(\eta-\cos\psi)} . \tag{25}$$

The beating of the amplitude α becomes large when the fixed point appears in the physical range $-\frac{1}{2} < \alpha < \frac{1}{2}$.[5] The condition for this is

$$|\chi| < \alpha + 1 . \tag{26}$$

For the space charge, the condition (26) implies

$-0.40 < \nu_v - \nu_H < 0.20.$

The Laslett linear tune shift is $\Delta\nu_v = -0.24$ and $\Delta\nu_H = -0.15$. This is calculated for a Gaussian beam and is about factor two larger than for a uniform beam. The octupolar tune spreads are also calculated to be 0.11 horizontally and 0.16 vertically.

Conclusion

At an intensity of 4.5×10^{12} ppp at an injection energy of 500 MeV, a rather large octupolar space charge coupling is expected. Further, a rather large linear tune shift and tune spread are expected. These are one of the possible causes of the beam loss at injection in the KEK proton synchrotron.

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