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IMPROVED ANALYTIC STARTING POINTS FOR BEAM MATCHING PROBLEMS TO BE SOLVED ON DIGITAL COMPUTERS*

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ABSTRACT

Digital computer codes using methods of least squares[1] and linear programming[2] can solve extraordinarily complicated beam matching problems involving many simultaneous parameters provided that they are provided with the starting points "close" to the desired solution. Estimates based on zero-emittance beams frequently are inadequate for this prupose. We here develop some thin lens formulae for finite-emittance beams to provide improved initial values to the variable parameters.

INTRODUCTION

The design of a beam transport system usually requires the specification of waists and beam widths at several locations. Unless care is taken, such specifications can be contradictory, resulting in the failure of the matching computer code to locate a solution. We here introduce a simple method to determine feasible beam profiles and the lens strengths and locations to achieve them.

ENVELOPE OPTICS

As is customary, we treat the beam as lying within an ellipse of constant area within the two dimensional phase space of x and x'. Where this ellipse is symmetrical with respect to the axes ("upright"), we have a waist for which the maximum displacement is $\bar{x}$, and the maximum slope is $\overline{x'}$. Let M be the 2X2 transfer matrix that provides the transformation of the beam envelope from point 1 to point 2:

$$\begin{pmatrix} \bar{x}^2 \\ \overline{x'}^2 \\ \bar{x}(\bar{x})' \end{pmatrix} = \begin{pmatrix} M_{11}^2 & M_{12}^2 & 2M_{11}M_{12} \\ M_{21}^2 & M_{22}^2 & 2M_{21}M_{22} \\ M_{11}M_{21} & M_{12}M_{22} & M_{12}M_{21}+M_{11}M_{22} \end{pmatrix} \begin{pmatrix} \bar{x}^2 \\ \overline{x'}^2 \\ \bar{x}(\bar{x})' \end{pmatrix} ; \quad (1)$$

note that $(\bar{x})'$, the divergence of the envelope, differs from $\overline{x'}$, the maximum slope. This is the same matrix that transforms an arbitrary column vector written as (x^2, x'^2, xx'); it is also the matrix that transforms the Twiss parameters $(\beta,\gamma,-\alpha)$ over the same interval as shown by Gourian.[3]

Virtual Waists

At any point, the envelope of the beam may be treated as having drifted from a waist through a drift length d to the point of observation.[4] In most cases this waist does not really exist due to the imposition of some optical element; we call such a waist a virtual waist.

We can think of the beam as originating at a waist of half width x_o. If M is the transfer matrix from this origin to the point of observation, then from Eq. 1, we see the following:

$$\bar{x}^2 = M_{11}^2\,\bar{x}_o^2 + M_{12}^2\,\overline{x_o'}^2 , \quad (2)$$

$$\overline{x'}^2 = M_{21}^2\,\bar{x}_o^2 + M_{22}^2\,\overline{x_o'}^2 . \quad (3)$$

The slope of the envelope, $(\bar{x})'$, is given by

$$\bar{x}(\bar{x})' = M_{11}M_{21}\bar{x}_o^2 + M_{12}M_{22}\overline{x_o'}^2 . \quad (4)$$

This envelope appears to issue from a virtual waist of half width $x_w = x_o x_o'/x'$ (5)

located upstream at a distance d, where

$$d = \bar{x}\,(\bar{x})' \,/\, (\overline{x'})^2 \quad (6)$$

Waist-to Waist Matching

In terms of the Twiss parameters, the most general transformation between a waist at point 1 and a waist at point 2 is

$$M = \begin{pmatrix} \sqrt{\beta_2/\beta_1}\,\cos\phi & \sqrt{\beta_1\beta_2}\,\sin\phi \\ -\sin\phi/\sqrt{\beta_1\beta_2} & \sqrt{\beta_1/\beta_2}\,\cos\phi \end{pmatrix} , \quad (7)$$

where ϕ is the (arbitrary) phase advance.[5]

Given that there exists a waist at point 1, the condition that there exist a waist at point 2 is:

$$\beta_1^2 = -M_{12}M_{22}/(M_{11}M_{21}) . \quad (8)$$

Waist-to-Waist Transfer by a Single Thin Lens

Let the beam system between point 1 and point 2 comprise a drift space of length a, a thin lens of focal length f, and a drift space of length b. We also introduce the lengths a' and b', measured from the focal points: a' = a-f; b' = b-f.

Applying Eq. 7 and Eq. 8 to this configuration, we find:

$$\beta_2/\beta_1 = b'/a' , \quad (9)$$

and

$$\beta_1\beta_2 = f^2 - a'b' . \quad (10)$$

The right hand side of Eq. 10, when set to zero, is recognized as the Newton image condition for zero-emittance optics. For the typical situation where the amplitude functions β are comparable in magnitude to the focal lengths and element spacings, this form shows clearly the difference between the conditions for waist-to-waist matching and those for providing a simple image.

We also observe that the ratio of the squares of the waist sizes is given by a length ratio whereas in zero-emittance optics, it is the first power -the magnification- that is given by a similar ratio.

We can express the focal length in terms of the two drift distances and the amplitude β at the lens using Eq. 9 and an expression from the next section:

$$f = (\beta_2 a - \beta_1 b)/(\beta_2 - \beta_1) , \quad (11)$$

$$\text{where } \beta_1 = \beta/2 \pm \sqrt{(\beta/2)^2 - a^2} \quad (12)$$

$$\text{and } \beta_2 = \beta/2 \pm \sqrt{(\beta/2)^2 - b^2} . \quad (13)$$

The choice of sign of the radicals refers to the possibility of two different waists as is shown below.

Before leaving this topic, we should mention that either or both of the drift distances a and b may be negative as may be the focal length.

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A GRAPHICAL METHOD

Waist Locus

It is well known that the envelope in a field-free region adjacent to a waist is a hyperbola (Eq. 2) that is asymptotic to the trajectories of maximum slope. If β_o is the amplitude function at the waist then the amplitude function β at a point d on either side of the waist is

$$\beta = \beta_o + d^2/\beta_o \quad . \tag{14}$$

This equation may be rewritten to yield the locus z and amplitude β_o given that the beam at the reference point z=0 (which might be a lens, a target, or some purely arbitrary point) has an amplitude β:

$$(\beta_o - \beta/2)^2 + z^2 = (\beta/2)^2 \quad . \tag{15}$$

This locus, which is seen to be a circle of radius $\beta/2$, centered at $(0, \beta/2)$ is shown in Fig. 1.

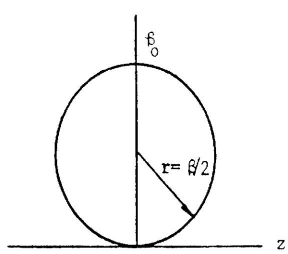

Fig. 1. Locus of β_o

If the reference point is in a field-free region then the waist may be either upstream or downstream of the point. There is a maximum distance $|z| = \beta/2$ beyond which there may not be a waist. For other locations, there are two waists that may be attained. If the reference point is taken to be at a thin lens, then it is possible for a waist to be on both sides of the point. The required focal length in this case is given by Eq. 11.

Two Reference Points with a Common Waist

Let us add a second point where we know the amplitude to be β_2. Let us use β_1 for the amplitude of the first reference point. We may in general have the loci shown in Fig. 2 where there are two possible amplitudes for the intervening waist:

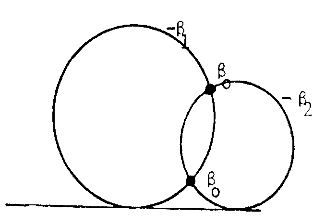

Fig. 2. Common Waist

Economy System

If the two loci in Fig. 2 are tangent then we have the situation where the drift distance between the two reference points, where the amplitudes are specified, is maximized. Moreover, if the two amplitudes are the same, we have

$$\beta_o = z = \beta/2 \tag{16}$$

which is the relationship for a periodic channel where minimum apertures are required for a given interelement spacing 2z.

Minimum Width

If the problem is to minimize the beam width at the second reference point, it is easy to show that such a condition is achieved when a waist precedes the second reference point as shown in Fig. 3. Although as pointed out in Banford[6], the excessive divergence of the beam from such a point makes this configuration less desireable than one where there is a waist at the second point.

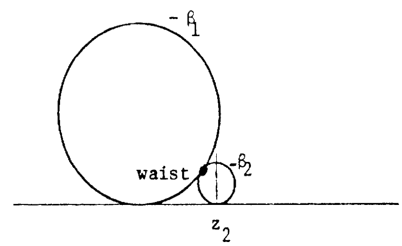

Fig. 3. Minimum Beam Width

Amplitude Function Locus

Consider the family of circles of varying radii that are all tangent to the z axis and that all pass through a common point. The curve tangent to all of these circles is the locus of Eq. 14.

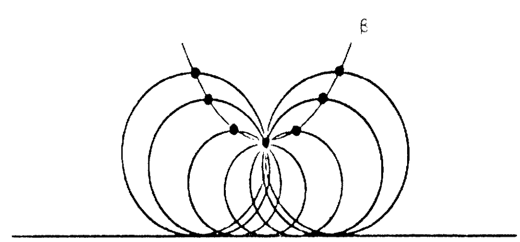

Fig. 4. Amplitude Function Locus

Diverging Lens

If we now assume the reference point to be a diverging lens, we can specify two waist locations and consider families of circles of the type in Fig. 4. for each point. The nearer point is taken to be a virtual waist, and circles through this point apply to the envelope upstream of the lens. The other point corresponds to the real waist produced by the lens, and circles through that point apply to the envelope downstream of the lens. Such a locus is shown in Fig. 5.

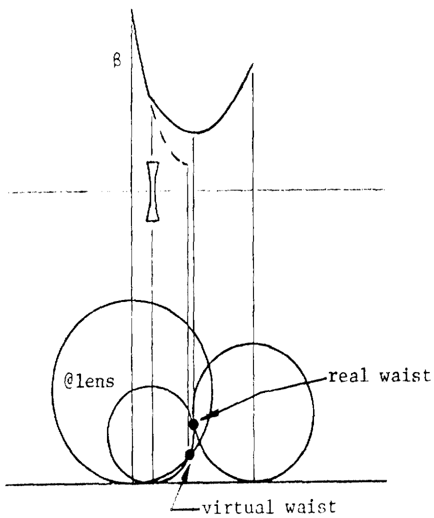

Fig. 5. Diverging Lens

Two Lens Variable Waist

Starting with a known source waist, it is possible with two lenses to produce a variable-size waist as shown in Fig. 6. However, the spacing between the lenses must not be much larger than β_1 if a reasonable range of sizes is to be available. For each possible output waist, the position of all waists and the amplitude at each lens is taken from the graph. Eq. 11 is then used to provide the two focal lengths.

For a given system with given apertures, the method provides a quick determination of the range of possible waist sizes.

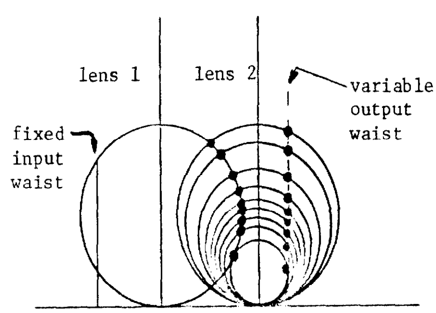

Fig. 6. Variable Waist

Complex System

We can next consider a transport system comprising of several lenses as sketched in Fig. 7 where two focusing lens and two defocusing lenses are used to transport a beam from one waist to another waist. Between each pair of lenses, we locate a real or virtual waist. Each waist, except the first and the last, lies on exactly two circles.

We start by drawing for each lens a circle whose diameter is the amplitude at that lens. The first and last must, of course, pass through the waists at the ends of the system at the appropriate amplitudes. The positions of the other waists, and if permitted, the positions of the lenses, may be adjusted until each circle properly meets its neighbor.

Eq. 11 is used to calculate the focal length for each lens given the locations of the (virtual) waists and the amplitude function at the lens. This, of course, must be done simultaneously for both the lateral and the vertical planes. However excellent starting conditions for a computer match may be obtained by performing this exercise with but limited accuracy. The main point of this exercise is to provide the computer code with a feasible problem.

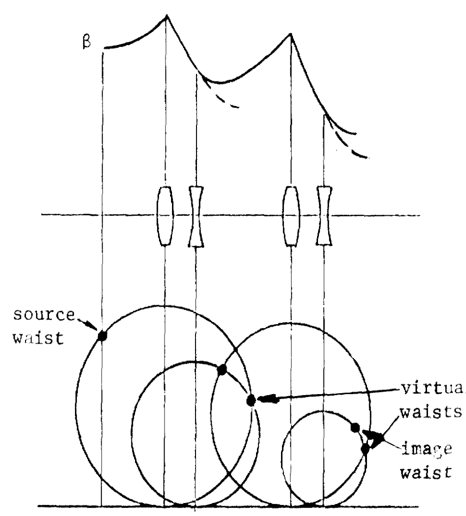

Fig. 7. Complex Systems

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