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JUMPING AN INTRINSIC DEPOLARIZATION RESONANCE IN SYNCHROTRONS

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We have derived a generalized version of the Froissart-Stora formula, which allows one to calculate the effect, on the polarization, of the finite Q-jump magnitude, ΔQ, in crossing rapidly an intrinsic depolarization resonance during acceleration.

Introduction

The possibility of accelerating polarized proton beams to high energies was demonstrated for the first time[1] at the Argonne ZGS in 1973. To accomplish this, the so-called intrinsic spin-resonances were rapidly passed through[1] by changing at proper times, during the acceleration cycle, the vertical betatron wave number, Q, of the Machine. Undoubtely, the major effort in the future will be focused on acceleration of polarized proton beams in the CERN/PS[2], AGS[3], FNAL booster[3] and KEK Proton Synchrotron[4]. Besides, acceleration of polarized electron beams came into begin recently[5] at the Bonn Synchrotron.

Since, in this context, it seems desirable to treat the effect, on polarization, of the finite Q-jump magnitude, ΔQ, in crossing rapidly an intrinsic depolarization resonance during acceleration, in the present paper we give the derivation of a generalized version of the well known Froissart-Stora formula[6]. This generalized formula is anticipated herebelow:

$$S_{z(t\to+\infty)} = 2\left[\sinh(p\pi)/\sinh(r\pi)\right]^2 - 1 \,. \qquad (1)$$

Here,

$$p = (1/2)(\Delta Q/2)^2/Q' \,, \quad \text{and} \qquad (2a)$$

$$r = p\left[1 + (\varepsilon/(\Delta Q/2))^2\right]^{1/2} \,. \qquad (2b)$$

In eqs. (2a, b), $Q' = \dot{Q}/\omega_0$ is the rate of change (per radian around the accelerator) of Q at the resonance crossing ($\dot{} \equiv d/dt$; ω_0 denotes the particle's angular velocity), ΔQ is the range of the Q-jump, $\varepsilon = \omega/\omega_0$ (ω is the width[6] of the resonance for a selected group of particles having the same amplitude of vertical betatron oscillations), and the assumption $S_{x(t\to-\infty)} = S_{y(t\to-\infty)} = 0$, $S_{z(t\to-\infty)} = 1$ is implicit. Eq. (1) is valid for any magnitude of ΔQ, and allows one to calculate the resulting depolarization in cases where the Froissart and Stora model is inadequate, namely when Teng's condition[7] is violated. We recall that Teng's condition demands that $\dot{Q}$ remains constant during the whole range of time $-(\omega_0\dot{Q})^{-1/2} < t < +(\omega_0\dot{Q})^{-1/2}$ where the resonance is effective, so that there exist cases where the applied range ΔQ may be too small for Teng's condition to be considered satisfied.

We also recall that the effective residue polarization can be calculated by averaging $S_{z(t\to+\infty)}$ over the vertical betatron amplitude distribution.

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Adiabatic following solutions before and after the passage of the resonance tail

Fig. 1 shows the direction of the polarization vector $\bar{S}$ (for a selected group of particles having the same amplitude of vertical betatron oscillations) at different

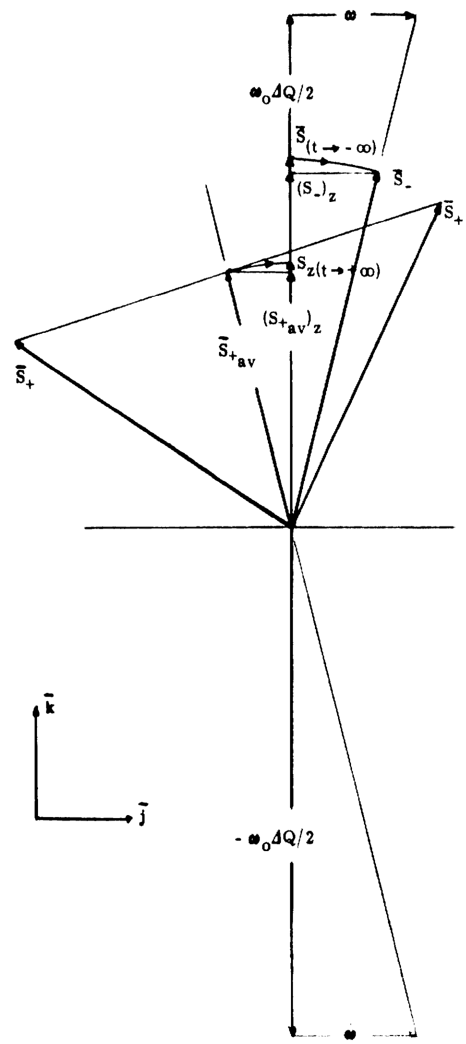

FIG. 1 - Vector diagram, in the (y, z) plane, showing the variations of $\bar{S}$ during the whole time interval $-\infty < t < +\infty$ in the case where a Q-jump is applied in the vicinity of an intrinsic spin-resonance. See text for the meaning of the symbols. $|\bar{S}_{(t\to-\infty)}| = |\bar{S}_-| = |\bar{S}_+| = 1$. $|\bar{S}_{+av}| = S_{z(t\to+\infty)}$.

times during the acceleration stages and the Q-jump stage. In the rotating reference frame, one may expect the following sequence of events to happen during normal acceleration, as the pulsed quadrupoles are subsequently turned on in the vicinity of the resonance, and eventually, i.e., at $t\to+\infty$, after the resonance is passed.

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Sufficiently far from resonance, $\bar{S}_{(t\to-\infty)}$ is oriented along the main field direction, $\bar{k}$. When approaching the resonance, the effective field $\omega\bar{j} + \dot{\chi}\bar{k}$ ($\dot{\chi} \gg \omega$), around which $\bar{S}$ precesses, starts turning slowly away from the vertical, and $\bar{S}$ will follow this change in orientation adiabatically[8]. Thus, before the occurence of the Q-jump pulse, the vertical component, S_z, of $\bar{S}$ is given by[8]

$$S_{z(\gamma < \gamma_{res})} = 1/\left[1 + (\omega/\dot{\chi})^2\right]^{1/2}, \qquad (3)$$

where $-\dot{\chi} = \omega_0 G(\gamma - \gamma_{res})$ is the detuning[6] ($\gamma = E/(mc^2)$ and G is the gyromagnetic anomaly).

Suppose now that when $\dot{\chi}$ becomes equal to $\omega_0 \Delta Q/2$, and, consequently, when

$$S_{z(\gamma < \gamma_{res})} = (S_-)_z \ , \qquad (3a)$$

with

$$(S_-)_z = p/r \qquad (4)$$

(see Fig. 1), the pulsed quadrupoles are switched on. Clearly $\bar{S}_-$ will start to precess around a rapidly varying effective field. Thus, after the resonance tail is passed through, the overall effect of this process is that it will lead to a polarization vector $\bar{S}_+$ precessing around the effective field $\omega\bar{j} - \omega_0(\Delta Q/2)\bar{k}$. Let be $\bar{S}_{+av}$ the time-average value of $\bar{S}_+$. $(S_{+av})_z$ will be expressed later, on the basis of a suitable model.

Eventually, i.e. over the remaining part of the acceleration cycle, $\bar{S}_{+av}$ undergoes an adiabatic process of following the direction of the effective field, and slowly aligns along the $\bar{k}$ direction as $\dot{\chi} \to -\infty$. Thus, we may write

$$S_{z(t\to+\infty)} = (S_{+av})_z \, r/p \ . \qquad (5)$$

Joining together of the adiabatic following solutions

We now look for a relation between $(S_{+av})_z$ and $(S_-)_z$ in order to derive a convenient asymptotic form for $S_{z(t\to+\infty)}$, i.e., eq. (1).

As a simple model which exhibits sweeping of $\dot{\chi}$ through zero, let

$$\omega = \text{constant} \qquad \text{and} \qquad \dot{\chi} = -\delta\tanh(\alpha t) \ , \qquad (6a, b)$$

where α and δ are assumed to be constant parameters. Clearly,

$$\left|\dot{\chi}_{t\to\pm\infty}\right| = \delta = \omega_0 \Delta Q/2 \ , \qquad (7a)$$

and

$$\left|\ddot{\chi}(0)\right| = \alpha\delta = \omega_0 \dot{Q} \ . \qquad (7b)$$

Once the model above is adopted, the object is to solve the conventional coupled equations of motion[6]

$$2i\dot{f} = \omega g \exp(i\chi) \ , \qquad 2i\dot{g} = \omega f \exp(-i\chi) \qquad (8a, b)$$

(which can be derived by considering the Schrödinger equation for a spin 1/2 particle interacting with an external field) written, as done above, in the rotating perturbing field approximation. ($gg^* + ff^* = 1$ is our normalization condition).

Thus, in the following, the time-average value of the transition probability,

$$(gg^*)_{+av} \equiv (1/2)\left[1 - (S_{+av})_z\right] \ , \qquad (9)$$

in the long-time limit will be calculated for a particle initially in the state given by eq. (4), by inserting eqs. (6) in eqs. (8) and solving these equations.

The first step consists of decoupling the spin-motion equations. We have, e.g., for g,

$$\ddot{g} - i\delta\tanh(\alpha t)\,\dot{g} + (\omega/2)^2 g = 0 \ . \qquad (10)$$

Eq. (10) is easily transformed into the equation

$$x^2(1-x)g'' + x\left[(1+ip)+(-1+ip)x\right]g' + (q/2)^2(1-x)g = 0 \qquad (11)$$

by the transformation

$$x = -\exp(2\alpha t) \ , \qquad (' \equiv d/dx \ ; \quad '' \equiv d^2/dx^2) \ . \qquad (12)$$

The dimensionless constants p and q are given by

$$p = \delta/(2\alpha) \qquad \text{and} \qquad q = \omega/(2\alpha) \ . \qquad (13a, b)$$

On introduction of the new function y by the substitution[9]

$$g = x^{\lambda} y \ , \qquad (14)$$

eq. (11) becomes[9]

$$x(1-x)y'' + \left[c-(a+b+1)x\right]y' - aby = 0 \ . \qquad (15)$$

The constants λ, c, a and b are

$$2\lambda_{\pm} = -ip \pm ir \ ; \quad c_{\pm} = 1 \pm ir \ ; \quad r = (p^2+q^2)^{1/2} \qquad (16a, b, c)$$

$$a_+ = -ip + ir \ ; \qquad b_+ = -ip \ ; \qquad (16d, e)$$

$$a_- = -ip \ ; \qquad b_- = -ip - ir \ . \qquad (16f, g)$$

Eq. (15) is the well known hypergeometric differential equation[10]. Thus, in the vicinity of x = -0 (i.e. at $t\to-\infty$) the solution of eq. (11) which satisfies the boundary condition (4) can be written[11] as follows:

$$g = D(-x)^{-ip/2 - ir/2} F(-ip-ir, -ip; 1-ir; x) \ . \qquad (17)$$

Here, D is the proper integration constant, namely,

$$DD^* = gg^*_{t\to-\infty} \equiv (gg^*)_- \equiv (1/2)\left[1-(S_-)_z\right] = (1 - p/r)/2 \qquad (18)$$

and F(-ip-ir, -ip; 1-ir; x) is a hypergeometric function (F(a, b; c; x) ≡ F(b, a; c; x); F(a, b; c; 0) = 1).

Note that solution (17) is consistent with the coupled nature of the original eqs. (8). This can be easily verified by considering separately both eqs. (8) at x=-0 (use the differentiation formula[12] for the F functions and remember that $ff^*_{t\to-\infty} = (1+p/r)/2$).

The asymptotic form for g as $t\to+\infty$ (i.e. $x\to-\infty$) can be found by making use of the appropriate linear transformation formula[13] for the F(-ip-ir, -ip; 1-ir; $x\to-\infty$) function appearing in eq. (17). Replacing in the resulting linearly-transformed[13] form of $g_{t\to+\infty}$ the F(a, b; c; 1/x) functions by unity and performing a fair amount of Γ-algebra, one finds

$$g_{t\to+\infty} = D(-x)^{ip/2 - ir/2}\left[\Sigma - (-x)^{ir}\sinh(p\pi)/\sinh(r\pi)\right], \qquad (19)$$

where

$$\Sigma = r\Gamma^2(-ir)\left[(r-p)\Gamma(-ip-ir)\Gamma(ip-ir)\right]^{-1} . \qquad (19')$$

Here, the Γ's are gamma functions.

Finally, we develop the squared modulus $gg^*_{t\to+\infty}$ and use eq. (18). We get for the time-average value of the transition probability, $(gg^*)_{+av}$,

$$2(gg^*)_{+av} \equiv 1-(S_{+av})_z = (1+p/r)s_-s_+ + (1-p/r)\left[\sinh(p\pi)/\sinh(r\pi)\right]^2 , \qquad (20)$$

where

$$s_\pm = \sinh\left[(r \pm p)\pi\right]/\sinh(r\pi) . \qquad (20')$$

In writing eq. (20) we have dropped an (expected) rapidly oscillating term which enters (its time-average value is zero) since we are interested only in the mean value of the residue polarization.

By some algebraic manipulation, eq. (20) may be written as[14]

$$(S_{+av})_z = (p/r)\left[2\sinh^2(p\pi)/\sinh^2(r\pi) - 1\right] . \qquad (21)$$

Thus it follows from eq. (5) that $S_{z(t\to+\infty)}$ is expressed by eq. (1).

Finally, we note that eq. (1) reduces to the Froissart and Stora formula[6] in the case where Teng's condition[7], i. e.

$$p \gg 2 \qquad (22)$$

is satisfied.

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14. The author is grateful to R. D. Ruth (BNL) for pointing this out.