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Crossing of Depolarization Resonances in Strongly Modulated Structures

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Summary

The present paper has two purposes .
- First, to give a general review of the theory of spin behavior in strongly modulated machines and to offer practical formulas for quick estimate of the depolarizing effect. Parts of this theory has already been published. (5)
- Then, by applying these formulas to Saturne 2, to show that this kind of structure is as well adapted to polarized beam acceleration as weak focusing machines.

1. Introduction

The beam polarization being defined as $S_z = \frac{n_+ - n_-}{n_+ + n_-}$ ($n_\pm$ are the numbers of particles with spin $\pm\frac{1}{2}$), it is well known (1) that after crossing a resonance, the final polarization is given by

$$S_{z\,final} = \left(2e^{-A^2} - 1\right) S_{z\,init} \qquad (1)$$

$S_{z\,init}$ is the initial beam polarization and is assumed to be ± 1 ; A^2 is proportional to the square of the strength of the depolarizing field components and is inversely proportional to the crossing speed. This speed is either related to the energy increase or to the tuning variation. In order to maintain a high beam polarization, it is necessary to work in either one of the following cases : $A^2 \ll 1$ or $A^2 \gg 1$.

Weak resonances ($A^2 \ll 1$)

This case is usually encountered in weak focusing lattices (ZGS type). The depolarizing fields are rather small (magnet edges and fringing fields). The tendency is to increase the crossing speed in order to prevent spin flip for large betatron amplitudes. An adequate theory for weak focusing has been extensively developed. (1), (2), (3)

Intermediate case ($A^2 \simeq 1$)

It is rather difficult to accelerate polarized particles in structures with intermediate focusing.

Strong resonances ($A^2 \gg 1$)

They appear in strongly modulated lattices (Saturne 2) (4), where strong transverse field components are encountered in quadrupole magnets. Then it is worthwhile decreasing the crossing speed to ensure adiabatic resonance crossing (complete spin flip) even for small betatron amplitudes.

In the following part of this paper, the quantity A^2 is written as

$$A^2 = \frac{\pi}{2} \frac{|\lambda|^2 |I_{Mn}|^2}{a} \qquad (2)$$

λ is explicitely given in part 2 ; I_{Mn} proportional to the depolarizing field components, is derived in part 3 ; the factor a is defined in part 4 where resonance crossing is considered.

2. Spin behavior through depolarizing fields

The spin vector $\vec{S}$ satisfies in the laboratory frame

$$\frac{d\vec{S}}{dt} = \frac{e}{m_o\gamma} \vec{S} \wedge \left[\vec{B} + G\left(\vec{B}_{/\!/} + \gamma \vec{B}_{\perp}\right)\right] \qquad (3)$$

with $|\vec{S}| = 1$ and where $\vec{B}_{/\!/}$ and $\vec{B}_{\perp}$ are respectively the longitudinal and transverse field components.
($G_{proton} = 1.7935$; $G_{deuteron} = -0.143$)

Let
$$\omega_p(\theta) = \frac{e}{m_o\gamma}(1+\gamma G)B_{z_o} = (1+\gamma G)\frac{R}{\rho_o}\omega_o \qquad (4)$$

$$\Delta\omega_p(\theta) = \frac{e}{m_o\gamma}(1+\gamma G)\Delta B_{z_o} \qquad (5)$$

$\omega_o = \frac{d\theta}{dt}$: revolution frequency ; R : mean radius ; ρ_o : bending radius ; B_{zo} : vertical component inside the magnet.

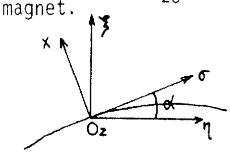

Fig. 1

Let $s = S_\eta + jS_\xi$; then s verifies

$$\frac{ds}{d\theta} = -j\frac{\omega_p(\theta)}{\omega_o} s - S_z e^{-j\alpha}\left[\lambda_x \frac{B_x}{B_{z_o}} + \lambda_\sigma \frac{B_\sigma}{B_{z_o}}\right] - js\frac{\Delta\omega_p(\theta)}{\omega_o} \qquad (6)$$

$$\lambda_x = (1+\gamma G) R/\rho_o \quad ; \quad \lambda_\sigma = -j(1+\ G) R/\rho_o$$

$$f(\theta) = -S_z e^{-j\alpha}\left[\lambda_x \frac{B_x}{B_{z_o}} + \lambda_\sigma \frac{B_\sigma}{B_{z_o}}\right] - j\frac{\Delta\omega_p(\theta)}{\omega_o} s$$

being the perturbation.

Equation without perturbation

$$\frac{ds}{d\theta} = -j\frac{\omega_p(\theta)}{\omega_o} s \qquad (7)$$

The solution can be written as

$$s = C\,S(\theta)\,e^{-j\bar{\omega}_p\theta/\omega_o} \qquad (8)$$

where $\bar{\omega}_p = (1+\gamma G)\omega_o = \frac{1}{\Theta}\int_\theta^{\theta+\Theta} \frac{\omega_p(\theta)}{\omega_o} d\theta$; Θ is the period of the structure ;
$S(\theta) = e^{j\left[\frac{\bar{\omega}_p}{\omega_o}\theta - \int_0^\theta \frac{\omega_p(\theta)}{\omega_o} d\theta\right]}$ is a periodical function ; $S(\theta+\Theta) = S(\theta)$; $|S(\theta)| = 1$

Equation including the perturbation

The solution of Eq. (6) can be obtained using the variation of constants method. It follows that

$$s = \left(s_o + C(\theta)\right) S(\theta)\, e^{-j\frac{\bar{\omega}_p}{\omega_o}\theta} \qquad (9)$$

with

$$C(\theta) = \int_0^\theta \frac{f(\xi)}{S(\xi)} e^{j\frac{\bar{\omega}_p}{\omega_o}\xi} d\xi \qquad (10)$$

3. Derivation of I_{Mn}

The transverse field components in dipole or quadrupole magnets can be written as

$$\begin{aligned} B_z(\theta,x,z) &= B_z(\theta,o,o) + K_\perp(\theta)(B\rho_o)x + \ldots \\ B_x(\theta,x,z) &= K_\perp(\theta)(B\rho_o)z + \ldots \\ B_\sigma(\theta,x,z) &= K_{/\!/}(\theta)(B\rho_o)z + \ldots \end{aligned} \qquad (11)$$

with $K_\perp(\theta) = \partial B_z/\partial x$ and $K_{/\!/}(\theta) = \partial B_z/\partial\sigma$ (periodical)
The transverse coordinates x, z of the particle are solutions of Hill equations and are expressed as follows $\begin{smallmatrix}x\\z\end{smallmatrix} = F_{\begin{smallmatrix}x\\z\end{smallmatrix}} e^{j(\nu_{\begin{smallmatrix}x\\z\end{smallmatrix}}\theta + \varphi_{\begin{smallmatrix}x\\z\end{smallmatrix}})} + c.c.$ (12)

where
$$F_{\begin{smallmatrix}x\\z\end{smallmatrix}} = \frac{1}{2}\sqrt{\frac{\beta_{\begin{smallmatrix}x\\z\end{smallmatrix}}\mathcal{E}_{\begin{smallmatrix}x\\z\end{smallmatrix}}}{\pi}}\, e^{j\left[R\int_0^\theta \frac{d\theta}{\beta_{\begin{smallmatrix}x\\z\end{smallmatrix}}} - \nu_{\begin{smallmatrix}x\\z\end{smallmatrix}}\theta\right]} \qquad (13)$$

$\nu_{\begin{smallmatrix}x\\z\end{smallmatrix}}$: wave numbers ; $\mathcal{E}_{\begin{smallmatrix}x\\z\end{smallmatrix}}$: transverse emittances.
For weak focusing structures, x and z are

$$\begin{smallmatrix}x\\z\end{smallmatrix} = \begin{smallmatrix}\hat{x}\\\hat{z}\end{smallmatrix}\cos(\nu_{\begin{smallmatrix}x\\z\end{smallmatrix}}\theta + \varphi_{\begin{smallmatrix}x\\z\end{smallmatrix}}) \quad \text{and} \quad F_{\begin{smallmatrix}x\\z\end{smallmatrix}} = \frac{1}{2}\begin{smallmatrix}\hat{x}\\\hat{z}\end{smallmatrix} \qquad (14)$$

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Vertical field components B_z

This component does not generate any depolarizing effect but modulates the precession frequency of the spin. The relevant term in Eq. (6) can be written as

$$\frac{dC}{C} = -j\,\frac{\Delta\omega_p(\theta)}{\omega_0} = -jR(1+\gamma G)K_\perp(\theta)\left[F_x(\theta)\,e^{j\nu_x\theta+\varphi_x} + c.c.\right] \qquad (15)$$

Its average vanishes since ν_x is not an integer.

Horizontal field components B_x

The perturbation function is

$$f(\theta) = -S_z\, e^{j\alpha}\,\lambda_x\,\frac{B_x}{B_{z_0}} \qquad (16)$$

It follows that

$$\frac{dC}{d\theta} = -\lambda_x \frac{S_z}{S(\theta)}\, e^{j(\theta-\alpha)}\, e^{j\gamma G\theta} K_\perp(\theta)\rho_0 \left[F_z(\theta)\, e^{j(\nu_z\theta+\varphi_z)} + c.c.\right] \qquad (17)$$

where $\rho_0\, e^{j(\theta-\alpha)} K_\perp(\theta)\,\frac{F_z(\theta)}{S(\theta)}$ and $\rho_0\, e^{j(\theta-\alpha)} K_\perp(\theta)\,\frac{F_z^*(\theta)}{S(\theta)}$ are periodical with period Θ and therefore can be expanded into Fourier series. Then,

$$\frac{1}{S_z}\frac{dC}{d\theta} = -\lambda_x \sum_{n=-\infty}^{+\infty}\left(|C_{Mn}|\, e^{j[(\gamma G+\nu_z+Mn)\theta+\varphi_z+\gamma_{Mn}]} + |D_{Mn}|\, e^{j[(\gamma G-\nu_z+Mn)\theta-\varphi_z-\delta_{Mn}]}\right) \qquad (18)$$

$$|C_{Mn}|\, e^{j\gamma_{Mn}} = \frac{\rho_0}{2\pi}\int_\theta^{\theta+2\pi} e^{j(\theta-\alpha)} K_\perp(\theta)\,\frac{F_z(\theta)}{S(\theta)}\, e^{-jMn\theta}\, d\theta \qquad (19)$$

$$|D_{Mn}|\, e^{j\delta_{Mn}} = \frac{\rho_0}{2\pi}\int_\theta^{\theta+2\pi} e^{j(\theta-\alpha)} K_\perp(\theta)\,\frac{F_z^*(\theta)}{S(\theta)}\, e^{-jMn\theta}\, d\theta \qquad (20)$$

$M = \frac{2\pi}{\Theta}$ in case of systematic depolarizing effects

$M = 1$ in case of random depolarizing effects

The resonance phenomenon occurs when $\gamma G \pm \nu_z + Mn = 0$

Considering the slowly varying part of $\frac{1}{S_z}\frac{dC}{d\theta}$ it follows that

$$\frac{1}{\sqrt{1-|C|^2}}\frac{dC}{d\theta} = -\lambda_x |I_{Mn}|\, e^{j(\varepsilon\theta+\psi_{Mn})} \qquad (21)$$

where $S_z^2 = 1-|s|^2 = 1-|C|^2$

$|I_{Mn}| = |C_{Mn}|$; $\psi_{Mn} = \varphi_z + \gamma_{Mn}$ if $\varepsilon = \gamma G + \nu_z + Mn$

$|I_{Mn}| = |D_{Mn}|$; $\psi_{Mn} = -\varphi_z + \gamma_{Mn}$ if $\varepsilon = \gamma G - \nu_z + Mn$

C_{Mn} and D_{Mn} are explicited in Eq. (19) and Eq. (20).

As an example, we find in Saturne 2 : $3.5 < \nu_z < 3.8$

$1 < \gamma < 4$; $M = 4$

Two systematic linear resonances $\gamma G = 8 - \nu_z$ and $\gamma G = \nu_z$ are crossed.

Longitudinal field components B_σ

All the results given for the horizontal components B_x are valid providing that $K_\perp$ is replaced by $K_{/\!/}$ and λ_x by λ_σ .

Closed orbit resonances

The transverse field along the closed orbit is

$$B_x = K_\perp(\theta)\,(B\rho_0)\, Z_{c.o.}(\theta) \qquad (22)$$

Eq. (18) can be written as

$$\frac{1}{S_z}\frac{dC}{d\theta} = -\lambda_x \frac{e^{j(\theta-\alpha)}}{S(\theta)}\, e^{j\gamma G\theta} K_\perp(\theta)\rho_0 Z_{c.o.}(\theta) = -\lambda_x \sum_{n=-\infty}^{+\infty}|H_n|\, e^{j[(\gamma G-n)\theta-\alpha_n]} \qquad (23)$$

where

$$H_n\, e^{-j\alpha_n} = \frac{\rho_0}{2\pi}\int_\theta^{\theta+2\pi} e^{j(\theta-\alpha)}\,\frac{K_\perp(\theta)}{S(\theta)}\, Z_{c.o.}(\theta)\, e^{jn\theta}\, d\theta \qquad (24)$$

The resonance phenomenon occurs when $\gamma G - n = 0$; σ_z being the mean squared root of $Z_{c.o.}(\theta)$ in the quadrupole lenses, one gets

$$|I_{Mn}|^2 = \langle |H_n|^2\rangle = \left(\frac{1}{2\pi}\frac{\rho_0}{R}\right)^2 \sum_{i=1}^{N_Q} (K_\perp \ell)_i^2\, \sigma_{z_i}^2 \qquad (25)$$

where $(K_\perp \ell)_i$ is the focusing strength of the i^{th} quadrupole in the structure and N_Q the number of quadrupoles.

The resonances $\gamma G = 2, \ldots, 7$ are encountered in Saturne 2 ; they are quite strong and generate a complete spin flip.

Stationary case. Resonance width

When ε does not vary, the solution of Eq. (18) is $C = |C|\, e^{j(\varepsilon\theta+\varphi)}$ where $|C|$ is a constant given by

$$|C|^2\left(1+\frac{\varepsilon^2}{\lambda^2 I_{Mn}^2}\right) = 1 \qquad (26)$$

I_{Mn}^2 is proportional to $\mathcal{E}_z$; $|I_{Mn}|^2 = |H_n|^2\,\mathcal{E}_z$. For a given distribution function of $\mathcal{E}_z$, the value of ε (or equivalent $\Delta E = \varepsilon E_0/G$ where E_0 is the rest energy) corresponding to a beam polarization $\langle S_z\rangle$ can be interpreted as a resonance width analogous to stop band width. Two different distributions are considered here .

- $f(\mathcal{E}_z) = \frac{1}{\mathcal{E}_{zMAX}}$ $\quad 0 < \mathcal{E}_z < \mathcal{E}_{zMAX}$ $\quad f(\hat{z}) = 2\,\hat{z}/\hat{z}^2_{MAX}$

then

$$\varepsilon = \pm\frac{\lambda H_n}{2}\langle S_z\rangle\sqrt{\frac{\mathcal{E}_{zMAX}}{1-\langle S_z\rangle}} = \pm\lambda H_n\langle S_z\rangle\sqrt{\frac{\sqrt{3}\,\sigma_{\varepsilon_z}}{4(1-\langle S_z\rangle)}} \qquad (27)$$

σ_{ε_z} being the mean square root.

- $f(\mathcal{E}_z) = \mu\, e^{-\mu\mathcal{E}_z}$ $\quad f(\hat{z}) = \frac{\hat{z}}{\sigma_z^2}\, e^{-\frac{\hat{z}^2}{2\sigma_z^2}}$; Rayleigh distribution corresponding to a Gaussian distribution relatively to z and $p_z = \alpha_z Z + \beta_z\frac{dZ}{d\sigma}$. Then,

$$\varepsilon = \pm\lambda H_n\sqrt{\frac{1}{2\sqrt{2}}\,\frac{\sigma_{\varepsilon_z}}{1-\langle S_z\rangle}} \qquad (28)$$

These two different models roughly lead to the same numerical results.

Resonance crossing

When γ or ν_z vary, Eq. (18) becomes

$$\frac{1}{\sqrt{1-|C|^2}}\frac{dC}{d\theta} = -|\lambda||I_{Mn}|\, e^{j(\int\varepsilon d\theta+\psi_{Mn}+\varphi_n)} \;;\; \lambda = |\lambda|\, e^{j\varphi_\lambda} \qquad (29)$$

here

$$\frac{d\varepsilon}{d\theta} = G\frac{d\gamma}{d\theta} \pm \frac{d\nu_z}{d\theta} = \frac{G\rho_0 R}{E_0/e}\frac{dB_{z_0}}{dt} \pm \frac{d\nu_z}{d\theta} = a \qquad (30)$$

Assuming a = constant, let $\tau = \sqrt{a}\,\theta$; then Eq. (29) can be written as

$$\frac{1}{\sqrt{1-|C|^2}}\frac{dC}{d\tau} = -\frac{|\lambda|}{a}|I_{Mn}|\, e^{j(\psi_{Mn}+\varphi_\lambda)}\, e^{j\tau^2/2} \qquad (31)$$

Using $C = \sin\varphi(\tau)\, e^{-j(\Phi(\tau)-\psi_{Mn}-\varphi_\lambda)}$ and $S_z = \cos\varphi(\tau)$, it finally leads to

$$\frac{d}{d\tau}\left(\sin\varphi\, e^{-j\Phi}\right) = -\frac{|\lambda|}{\sqrt{a}}|I_{Mn}|\, e^{j\tau^2/2}\cos\varphi \qquad (32)$$

The solution is (1) :

$$\cos\Delta\varphi = 2\, e^{-\frac{\pi}{2}\frac{|\lambda|^2|I_{Mn}|^2}{a}} - 1 \qquad (33)$$

4. Numerical results for Saturne 2

Betatronic linear systematic resonances

	A^2		$\cos\Delta\varphi$	ΔE	
$\gamma G = \nu_z$	1208 $\hat{z} = 2.13$ cm	267 $\hat{z} = 1.$ cm	-1	± 67 Mev $\hat{z} = 2.13$ cm	± 31.5 Mev $\hat{z} = 1.$ cm
$\gamma G = 8 - \nu_z$	742 $\hat{z} = 1.95$ cm	195 $\hat{z} = 1.$ cm	-1	± 52.5 Mev $\hat{z} = 1.95$ cm	± 27 Mev $\hat{z} = 1.$ cm

The resonance width ΔE has been computed for $\langle S_z\rangle = .99$

Closed orbit resonances

	A^2	$\cos\Delta\varphi$	ΔE
$\gamma G = 2$	10.4	.1	± 8.7 Mev
$\gamma G = 3$	18.5	.1	± 11.6 Mev
$\gamma G = 4$	29.0	.1	± 14.5 Mev
$\gamma G = 5$	41.7	.1	± 17.0 Mev
$\gamma G = 6$	56.7	.1	± 20.4 Mev
$\gamma G = 7$	74.1	.1	± 23.2 Mev

The resonance width ΔE has been computed for σ_z = 4 mm.
This analytical approach has been checked by means of a computer program dealing with matrix transport of spin.

Other resonances

Non systematic linear components as well as non linear components are not strong enough to generate any depolarizing effect ($A^2 \ll 1$). These numerical results show that one should be able to accelerate polarized beams in a strongly modulated structure without any correction.

5. Conclusion

The generally accepted idea that strong focusing is not well adapted to polarized beam acceleration does not seem to apply to Saturne 2. The choice of strong focusing had been made because of its intrinsic advantages when compared to weak focusing. (4) The fact that strong focusing and polarized beam acceleration can be made consistent with each other is quite promising. The first polarized beam at Saclay is scheduled next october 79.

References

1. M. Froissart, and R. Stora
Dépolarisation d'un faisceau de protons polarisés dans un synchrotron
Nuclear Instruments and Methods, 2(1960), 297-305

2. L.G. Ratner, and T.K. Khoe
Acceleration of polarized protons in the zero gradient synchrotron (ZGS), IEEE Transactions on Nuclear Science (1973), Vol. NS-20, Nb 3, 217-220

3. L.C. Teng
Depolarization of a polarized proton beam in a circular accelerator, NAL FN-267-0100 (Nov. 74) and NAL FN-267 A-0100 (Jan. 75)

4. H. Bruck, J.L. Laclare, G. Leleux and Germa Group
Saturne 2, Proposal of renovated proton facility at Saclay, IX Conference on High Energy Accel. Stanford, Cal. (1974)

5. Kh. A. Simonian
Variation of beam polarization in circular accelerators, VII International Conference on High Energy Accelerators, Yerevan (1969)253-264

6. E. Grorud, J.L. Laclare, G. Leleux
Résonances de dépolarisation dans Saturne 2
Rapport interne GOC-GERMA 75-48/TP-28 (1975)
DSS-CEN/Saclay

7. E. Grorud, J.L. Laclare, G. Leleux
Résonances de dépolarisation dans Saturne 2
(résonances systématiques d'ordre supérieure à 1)
Rapport interne LNS-020, LNS_CEN/Saclay.