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FLUID AND VLASOV STABILITY OF INTENSE ION BEAMS IN PERIODIC CHANNELS

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Summary

Transverse beam behaviour of an intense beam through long periodic systems is reviewed critically. In view of the uncertainty about the proper choice of a distribution function the applicability of a hydrodynamical approach is discussed. Starting from a laminar beam, structure resonances and their shift by finite temperature are introduced as hydrodynamic features. The Vlasov treatment brings in additional structure resonances and entirely new features, like velocity space instabilities, if a loss-cone distribution (for instance K-V distribution) is chosen. The relevance of these instabilities for beam transport considerations is briefly discussed.

1. Introduction

The application of accelerators for heavy ions as drivers for inertial confinement fusion has necessitated the study of beam transport phenomena in a regime where space charge effects are so strong that single particle effects are dominated by coherent oscillation phenomena. A theoretical description of the transverse beam behaviour requires knowledge of the distribution function in four-dimensional phase space as starting point for a stability analysis. The question what the velocity distribution in a beam from a thermal source looks like is difficult to answer, which makes stability analysis difficult. While emission at a thermal source occurs with local Maxwellian velocity distribution the actual velocity distribution at a position further downstream is modified by finite (transverse) geometry effects leading to non-Maxwellian modifications of the distribution function. The highly artificial K-V (microcanonical) distribution is the only case for which the problem of transport through a periodic channel has been solved analytically [1].

2. Vlasov and Hydrodynamical Approaches

Lawson [2] has shown that optical and hydrodynamical descriptions are equivalent if envelope oscillations of a uniform beam are considered (in paraxial approximation, which is assumed here). A complete description of a beam as collisionless nonneutral plasma requires the use of the Vlasov equation. This equation is difficult to solve in the beam case and one may ask under what conditions it can be replaced by the much simpler hydrodynamic equations, which govern local macroscopic quantities obtained by averaging over the velocity space. Since the hydrodynamical approach is also more amenable to direct physical interpretation, it would be desirable to use it whenever possible.

For a formal derivation [3] we start from the Vlasov equations for electrostatic oscillations

$$\frac{\partial f}{\partial s} + (\underline{v}\cdot\underline{\nabla})f - (\underline{K} - \frac{q}{Mv^2}\underline{E})\nabla_v f = 0 \quad (1)$$

$$\underline{\nabla}\cdot\underline{E} = 4\pi q\int f dv_x dv_y \quad (2)$$

where the linear external focussing force is given by

$\underline{K} \equiv \{\kappa_x(s)x,\ \kappa_y(s)y\}$ ($\kappa_{x,y}(s)$ is $\frac{\pm B'}{(B\rho)}$ for quadrupoles and $\left(\frac{B}{2(B\rho)}\right)^2$ for solenoids in the co-rotating Larmor frame) and time is replaced by the distance $s = vt$, with v the longitudinal velocity. Averaging of (1) over the velocity space yields

$$\frac{\partial n}{\partial s} + \underline{\nabla}\cdot(n\underline{V}) = 0 \text{ (continuity equation)} \quad (3)$$

$$n\frac{\partial \underline{V}}{\partial s} + n(\underline{V}\cdot\underline{\nabla})\underline{V} + \frac{1}{M}\underline{\nabla}\cdot\underline{P} - \left(\frac{q}{Mv^2}\underline{E} - \underline{K}\right)n = 0 \text{ (momentum transport)} \quad (4)$$

with the density $n \equiv \int f dv_x dv_y$, the fluid velocity $\underline{V} \equiv n^{-1}\cdot\int \underline{v} f dv_x dv_x$ and the pressure tensor components $P_{ij} \equiv M\int(v_i - V_i)(v_j - V_j) f dv_x dv_y$, which satisfy an equation of state, with d/ds a total derivative,

$$\frac{d}{ds}P_{ij} + \sum_k \frac{\partial}{\partial x_k}Q_{ijk} + \sum_k\left(P_{ij}\frac{\partial V_k}{\partial x_k} + P_{jk}\frac{\partial V_i}{\partial x_k} + P_{ik}\frac{\partial V_j}{\partial x_k}\right) = 0 \quad (5)$$

The third moments $Q_{ijk} \equiv \int(v_i - V_i)(v_j - V_j)(v_k - V_k)\cdot f dv_x dv_y$, which describe the heat flow, are again connected with fourth moments, and so on. The hydrodynamical approximation requires that Q_{ijk} be small so that it can be neglected in (4) and we obtain a closed set of equations. This is strictly justified only if f is quadratic in $(v_x - V_x)$ and $(v_y - V_y)$, hence heat flow is suppressed. In general, thermal effects can be described by the pressure term in an average sense only. A beam close to the space charge limit, however, may support electrostatic oscillation much faster than the transverse "thermal" oscillation ($\omega \gtrsim \omega_p >> \sigma$, with ω_p plasma frequency and σ transverse tune). Hence, heat flow is negligible for fluctuations on such a time scale.

3. Matched Beam Solutions

Besides stability considerations, the entrance conditions of the beam have to be matched to a given transport channel to permit optimum transmission. This is a straightforward problem only in a continuous focussing channel, or in a periodic channel if it suffices to follow the envelope motion so that one can define a solution with the same period as the channel. In case of nonuniform space charge density and periodic focussing the envelope is inadequate. We characterize the following situations:

Continuous Solenoidal Focussing ($\kappa_x = \kappa_y = 1/(4\rho^2) = \sigma_o^2$)

From a theoretical point of view we can use an arbitrary function $f_o(H)$ of the Hamiltonian as constant of motion in four-dimensional phase space and obtain a stationary solution of (1), provided that we have solved Poisson's equation selfconsistently according to

$$\nabla^2\phi = -4\pi q\int f_o(H(\phi))dv_x dv_y \quad (6)$$

In this time-independent case the hydrodynamic equation (4) leads to an equivalent integral equation if we define a temperature according to

$$P(r) = n(r)kT(r) \quad (7)$$

and choose T(r) appropriately. It is easy to verify that

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in hydrodynamic terms the K-V distribution $f_o \sim \delta(H-H_o)$ is a fluid with uniform density, isotropic pressure $P_{xx} = P_{yy} = P(r)$ and a temperature that drops quadratically from the center of the beam. The Maxwellian beam with $f_o \sim \exp(-H/kT)$ can be obtained from equations (2) - (4), if we set $T(r) \equiv T$.

Periodic Focussing

If we allow for a periodic variation of the focussing strength $\kappa = \kappa(s)$ in a solenoid system the only known rigorous solution of equ.(1) is the K-V distribution [4] which leads to uniform density and an envelope subject to the well-known nonlinear equation

$$a'' = -\kappa(s)a + \frac{\varepsilon^2}{a^3} + \frac{2q^2N}{Mv^2a} \qquad (8)$$

where ε is the emittance and N the line density. From our observations in section 2 it is evident that the K-V solution has zero heat flow Q and can be looked at as hydrodynamic flow with isotropic pressure. Hence, equ.(8) can be derived also from the hydrodynamic equations (2) - (5) with $V_r = ra^{-1}\,\partial a/\partial s$ and this establishes the equivalence of emittance and pressure rsp. temperature:

$$\varepsilon^2 = \frac{2k}{M}\, a^2(s)\, T(0,s) \equiv \text{const.} \qquad (9)$$

which expresses the constancy of the product of beam area and peak temperature $(T(r,s) = T(0,s)(1-r^2/a^2))$. A matched beam is then defined as solution of equ.(8) with the same periodicity as $\kappa(s)$.

If we allow for nonuniform density, solution of equation (1) becomes extremely difficult. Definition of an invariant quantity, like emittance, is no longer straightforward, since the density in each two-dimensional projection of the phase space is nonuniform. We want to investigate some properties of the hydrodynamic approximation. For a rotationally symmetric beam we rewrite equations (2) - (5) in cylindrical coordinates and obtain with $P_{r\theta} = 0$

$$\frac{d}{ds}P_{rr} + 3\frac{\partial V_r}{\partial r}P_{rr} + \frac{V_r}{r}P_{rr} = 0$$

$$\frac{d}{ds}P_{\theta\theta} + \frac{\partial V_r}{\partial r}P_{\theta\theta} + \frac{3}{r}V_rP_{\theta\theta} = 0 \qquad (10)$$

$V_r(r,s)$ is no longer linear in r, hence we obtain anisotropic pressure, and with the continuity equation we find the adiabatic law

$$\frac{d}{ds}\left(\frac{P_{rr}P_{\theta\theta}}{n^4}\right) = 0 \qquad (11)$$

which can be expressed also in terms of an anisotropic temperature $T_{ii} = (nk)^{-1}P_{ii}$. Evaluation at the beam center, where $V_r = 0$, hence $d/ds = \partial/\partial s$, yields

$$\left[\frac{T_{rr}T_{\theta\theta}}{n^2}\right]_{r=0} \equiv \text{const.} \qquad (12)$$

as generalized version of condition (9). Equation (12) gives a macroscopic quantity which is constant along the channel and replaces the concept of constant emittance in this more general situation. We observe that radial beam motion is not self-similar in general, i.e. the radial profiles of all macroscopic quantities are changing with s.

Validity of the adiabatic law is based on negligible heat flow. As we mentioned above, this is a reasonable approximation, if beam fluctuations induced by the periodically varying focussing are rapid compared with thermal motions. Therefore we suggest as nontrivial application of the hydrodynamical equations the regime of strong tune depression by space charge and small phase advance of a particle per focussing period ($\sigma'/\sigma'_o \ll 1$ and $\sigma' \ll \pi$).

4. Stability Considerations

The question of stability of a given beam flow appropriately matched to a transport channel requires investigation of all possible eigenmodes. The stability result may depend very sensitively on the choice of the unperturbed distribution function [5]. In view of our uncertainty what a realistic unperturbed distribution should be it is of interest to first inquire in the stability properties of the hydrodynamical model, starting from the simplest case.

Laminar Flow

If thermal effects are neglected we obtain laminar or Brillouin flow, which is described by the hydrodynamic equations with $P \equiv 0$. For $\kappa(s) \equiv \kappa_o$ external focussing and space charge defocussing are equal. The matched radius is $a_o = (2q^2N/(Mv^2\kappa_o))^{1/2}$ and the plasma frequency $\omega_p \equiv 4\pi q^2 n/(Mv^2)$ is given by $\omega_p^2 = 2\kappa_o$. Small oscillations $E^{(1)} \sim e^{i\omega s}$ obey

$$\omega^2(\underline{\nabla}\cdot\underline{E}^{(1)}) = \frac{4\pi q^2}{Mv^2}(\underline{\nabla}\cdot(n_o\underline{E}^{(1)})) \qquad (13)$$

which yields the dispersion relation

$$\omega^2 = \omega_p^2 \quad \text{for body oscillations } (n^{(1)} \neq 0 \text{ for } r^2 < a_o^2)$$

$$\omega^2 = \omega_p^2/2 \text{ for surface oscillations } (n^{(1)} = 0 \text{ for } r^2 < a_o^2) \qquad (14)$$

In a periodic solenoid channel the amplitude of small oscillations is found subject to the equation

$$A''(s) + \alpha(s)\,A(s) = 0 \qquad (15)$$

with $\alpha(s) \equiv \kappa(s) + 1/2\,\omega_p^2(s)$ for body oscillations and $\alpha(s) \equiv \kappa(s)$ for surface oscillations, where $\omega_p^2(s) \equiv 4q^2N/(Mv^2a^2(s))$ is defined by a periodic envelope solution of equ.(8) with $\varepsilon = 0$, and $A \equiv \int (x/a(s))^\ell E_x^{(1)}dxdy$ is any nonvanishing moment of the electric field perturbation. Equ.(15) is of Hill's type with solutions $A = e^{i\omega s}\tilde{A}(s)$, where $\tilde{A}(s)$ has the period of $\kappa(s)$. It can be replaced by the simpler Mathieu equation if we consider a small harmonic variation $\kappa(s) = \kappa_o + \kappa_1\cos(\lambda s)$, hence for body oscillations $\alpha(s) = 2\kappa_o + \kappa_1(\lambda^2 - 4\kappa_o)/(\lambda^2 - 2\kappa_o)\cdot\cos(\lambda s)$. The harmonic content in $\alpha(s)$ can give rise to parametrically driven structure resonances if the frequency λ is twice the natural frequencies ω_p or $\omega_p/\sqrt{2}$. As usual, we express the stop bands in terms of σ'_o, zero intensity phase advance per period of the structure,

$$|\sigma'_o - \pi| \lesssim \frac{\pi}{4}\frac{\kappa_1}{\kappa_o} \text{ and } \left|\sigma'_o - \frac{\pi}{\sqrt{2}}\right| \lesssim \pi\frac{\sqrt{2}}{24}\frac{\kappa_1}{\kappa_o} \qquad (16)$$

for the surface and body oscillations, respectively; corresponding growth rates are $\sqrt{2}/8 \cdot \kappa_1/\kappa_o \cdot \omega_p$ and $1/12 \cdot \kappa_1/\kappa_o \cdot \omega_p$.

Finite Temperature (Hydrodynamic) Effects

While a cold beam allows for only two natural oscillation frequencies, ω_p and $\omega_p/\sqrt{2}$, thermal effects remove this degeneracy. For simplicity we have investigated the uniform density beam with finite pressure and find the following hydrodynamical eigenfrequencies $\omega_{m,j}$.

1. Envelope modes, $\omega^2_{0,1} = \omega^2_p + 4\sigma^2$ and $\omega^2_{2,0} = \omega^2_p/2 + 4\sigma^2$, with $\phi_{0,1} = Ae^{i\omega s}r^2$ and $\phi_{2,0} = Ae^{i\omega s}r^2 \cos 2\theta$ $(r \leqq a)$

2. Azimuthally symmetric (m = 0) nonuniform body oscillations, $\omega^2_{0,j} = \omega^2_p + 4\chi_j j^2\sigma^2$, with χ_j slightly dependent on j (χ_j = 1.38, 1.44, 1.47, ... for j = 2, 3, 4...) and $\phi_{0,j} = Ae^{i\omega s}(r^{2j} + b_1 r^{2(j-1)} + \dots b_j)$ $(r \lesssim a)$

Here, σ is defined as $(\kappa_0^2 - \omega_p^2/2)^{1/2}$ and easily identified as single particle tune, hence $\sigma = \sigma_0 = \kappa_0$ for zero intensity and $\sigma = 0$ for laminar flow. We observe that the envelope mode $\phi_{2,0}$ was the only nonsymmetric mode we have found from the hydrodynamic equations. We conclude that higher order nonsymmetric modes require the effect of heat flow, which is suppressed in the hydrodynamical model. The nonuniform symmetric body oscillations are characterized by density ripples inside the beam and anisotropic pressure $P_{rr} \neq P_{\theta\theta}$.

Again periodically varying focussing allows for structure resonances. Since finite temperature increases the mode frequencies, we obtain stop-bands that are shifted towards shorter focussing periods (i.e. smaller σ_0'), depending on m, j. In addition to the parametric case there may also appear structure resonances with integral or sum or difference relations. In case of growth, density fluctuations (j > 1) are accompanied by fluid velocity fluctuations which reduce beam quality.

Non-hydrodynamic Effects

The Vlasov treatment based on a distribution of particles in configuration and velocity space extends the possible electric field eigenmodes and allows, in principle, for a higher dimensionality of eigenfrequencies than the fluid treatment. Gluckstern [6] has derived the dispersion relation for the K-V distribution with continuous solenoid focussing. Despite of the unrealistic character of this distribution it is interesting to look at the eigenfrequencies $\omega^{(n)}_{m,j}$ with m the azimuthal and j the radial mode numbers (m,j=0,1,2...) and n=1... j+(m+1)/2 for odd m, n=1...j+m/2 for even m.

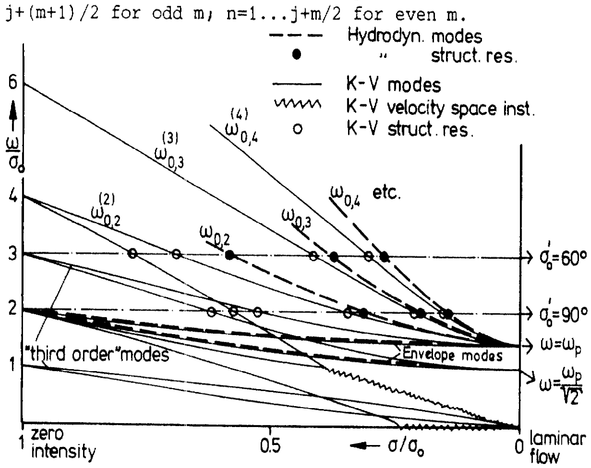

Fig.1: Eigenfrequencies (ω) of uniform beam in a long solenoid in hydrodynamical and Vlasov description (K-V case). Location of structure resonances (parametric, if $2\omega=\lambda$) with weak periodic variation of focussing is indicated for $\sigma_0'=60°, 90°$. With $\sigma_0<60°$ the envelope and "third order" resonances disappear.

We first remark that the eigenfrequencies of the hydrodynamical approach (for the uniform beam) agree with the K-V results for the envelope oscillations; in case of the azimuthally symmetric nonuniform body oscillations we obtain identical eigenfunctions, while the eigenvalues agree, if σ^2 is small compared with ω_p^2 so that oscillations are adiabatic and the equation of state is justified. This supports that the hydrodynamical description of nonuniform beams is adequate if fluctuations are rapid compared with the single particle tune σ.

The new phenomena that arise with the microscopic Vlasov treatment can be summarized as:

1. Additional structure resonances The K-V distribution allows for nonsymmetric electric field perturbations to arbitrary m and hence a higher dimensionality of structure resonances. Recently, parametric resonance of a third order mode (m = 3, j = 0; see Fig.1) has led to the suggestion $\sigma_0' \leq 60°$ to avoid it [7]. Since this mode depends on the presence of heat flow (a nonhydrodynamic effect), it is unclear how harmful it would be for a distribution quite different from the K-V.

2. Velocity space instabilities. Independent of interruptions in the focussing strength the K-V distribution exhibits unstable modes with $\mathrm{Re}\,\omega \to 0$ and $\mathrm{Im}\,\omega \to 0$ if $\sigma \to 0$ [1]. The loss-cone in the energy distribution is responsible for the occurrence of oscillations, which carry negative signal energy defined by $\overline{U}(t) = (8\pi)^{-1}[d(\omega E)/d\omega]_{\omega=\omega^{(n)}_{mj}} \overline{E}^2$ and yield internally driven instabilities by coupling with positive energy modes [8]. Onset of these K-V instabilities occurs if $\sigma/\sigma_0 \lesssim 0.39$ [1]. Eigenmode calculations for different distribution functions indicate that it suffices to partially fill up the K-V loss-cone in order to make these instabilities insignificant [5]. Hence, there are strong reasons to suppose that beams from a thermal source are not affected by this mode.

3. Landau damping The stabilizing effect of phase mixing may be considerable, if there is a sufficiently broad distribution of single particle frequencies in the unperturbed beam. The K-V beam has no Landau damping, and for other beams the effect can perhaps be investigated through computer simulation.

5. Conclusion

Hydrodynamical equations with anisotropic pressure can be a reasonable extension of the simple envelope equations (which are also hydrodynamical), if the heat flow term is negligible. This appears justified for uniform density in the beam (as the envelope equations require); or, with nonuniform density, if oscillations are rapid on the time scale of thermal motion, like in a weakly nonlaminar beam (rsp.strong tune depression). In this limit structure resonances of the hydrodynamic plasma oscillations require that σ_0' be sufficiently below the value $\pi/\sqrt{2}$, where the resonance bands of cold plasma oscillations are accumulated. For $\sigma_0'<90°$ one also avoids the stop-band of the envelope mode, but further study is necessary to determine whether higher order mode resonances are of importance. The numerous additional resonances of the K-V analysis must be judged similar, though at least a third order mode with large growth rate can be avoided if $\sigma_0'<60°$ [7]. Velocity space instabilities (which call for $\sigma/\sigma_0 \gtrsim 0.39$ in the K-V case) are inappropriate to define a threshold for tune depression, because a loss-cone distribution is unrealistic in this connection. The question whether stability considerations set a pronounced lower limit in σ/σ_0 at all should be investigated by computer simulation. Non-paraxial effects may have to be included to get reliable results.

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