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STABILITY OF INTENSE TRANSPORTED BEAMS*

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Summary

At the previous National Accelerator Conference, the transport of intense ion beams, with particular reference to Heavy Ion Fusion, was analyzed by finding matched solutions of the coupled envelope equations.[1] This work established relations between lattice structure, beam dimensions and space-charge tune depression as a function of intensity. In this paper we report on an investigation of the stability of the K-V distribution in transport by a periodic quadrupole system, a generalization of Gluckstern's analysis for a continuous solenoid.[2] The results are presented and compared with simulation computations for a particular case; the results provide a prediction of maximum transportable current without degradation of emittance due to instability.

Method[3]

The K-V distribution is unique in that it permits specification of a stationary state in the presence of variable linear external forces plus space charge forces. Since the motion of individual ions is governed by linear forces, a perturbed solution of Vlasov's equation can be usefully written as an integral of the perturbing forces along the unperturbed trajectories. The forces, in turn, are determined by Poisson's equation in the two transverse dimensions; integration of the perturbed distribution function over the transverse momentum variables then leads to an integro-differential equation for the perturbed potential of the form:

$$\frac{1}{a^2}\frac{\partial^2 V}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 V}{\partial y^2} = -\frac{Q}{\pi\varepsilon ab}\int_{-\infty}^{s} ds' \left[\frac{\partial}{\partial\psi_x'} + \frac{\partial}{\partial\psi_y'}\right]$$

$$\int_0^\infty dp^2 \delta'[p^2 - (1 - x^2 - y^2)] \int_0^{2\pi} d\theta\, V(x', y', s') \quad (1)$$

where $x' \equiv x(s') = x(s)\cos[\psi_x(s') - \psi_x(s)]$

$$+ p(s)\cos\theta \sin[\psi_x(s') - \psi_x(s)]$$

$$y' \equiv y(s') = y(s)\cos[\psi_y(s') - \psi_y(s)]$$

$$+ p(s)\sin\theta\sin[\psi_y(s') - \psi_y(s)]$$

$$\psi_{x,y}(s) = \int^s \frac{ds}{\beta_{x,y}}$$

$Q = \frac{4Nq^2}{A\beta^2\gamma^3} r_p$, where N = number of particles per unit length of beam

$\pi\varepsilon$ = emittance (assumed equal in both planes)

a,b = half width of the beam in x and y, respectively.

$\beta_{x,y}$ and a, b are periodic functions of s, determined as in reference (1).

In spite of the formidable appearance of eq'n (1), a brief inspection shows that the solutions for V(x,y,s) are mixed finite polynominals in x and y with coefficients that are functions of s. Eq'n (1) then reduces to a set of linear differential equations involving the coefficients. Finally, a numerical integration of these equations through one period using the appropriate a(s) and b(s) leads to a matrix, the eigen-values of which determine whether the motion is stable or unstable in the presence of that form of perturbation.

Results

We use a terminology in which an n^{th} order perturbation is one in which n is the highest power appearing in the perturbed potential; it is further called even or odd according to whether even or odd powers of y occur. Thus, for example, $V = Ax^4 + Bx^2y^2 + Cy^4$ is "fourth order even" and $V = Ax^2y + By^3$ is "third order odd". In practice, the number of equations for the coefficients increases so rapidly with order that we have not gone beyond sixth order. A finer grained structure to the perturbation is in fact not very interesting in view of the doubtful relation of the K-V distribution to a real beam.

The general character of the results is that instability occurs for all modes in finite ranges of intensity such that the frequencies of the modes pass through a rational relation to the Fourier components of the β-functions. In addition, for the even order modes a threshold is reached above which the motion is unstable for all higher intensities. For reasons not fully understood, these thresholds occur at precisely the same tune depression as for the continuous solenoid and the growth rates as functions of further tune depressions are precisely the same. Fig's 1 and 2 show the unstable regions for fourth and sixth order even perturbations for a FODO lattice with various zero-intensity phase advances per period.

The second order even perturbation ($V = Ax^2 + By^2$) corresponds to integrating the envelope equations with a slight initial mis-match. This mode is unstable at an intensity which depresses the phase advance per cell to 90° if the zero intensity phase advance per cell is greater than 90°. On the basis of this result we feel that a transport channel for high intensity beams should be designed for less than 90° at zero intensity.

Comparison with Simulation Computations

In parallel with the analytic work, extensive simulation computations have been carried out by Haber.[4] We find qualitative agreement for the onset of the extended region of instabiltiy but, since many modes are unstable in this region, a quantitative comparison is not possible. However, in a different parameter range, Haber found an instability which we were able to identify as an isolated third order structure resonance. From the perturbed distribution function for the mode, expressions for the growth of various moments of the distribution and the distortion of the phase space boundaries were derived and compared, period by period, with the simulation results. Surprisingly good agreement was found for the growth rate, the relative magnitudes of the moments and the boundary distortions, the only empirical parameters being the effective initial amplitudes of the odd and even modes. Figure 3 illustrates the development of this instability as the intensity is increased, the larger of the two

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180-deg. modes being identifiable with the instability observed by Haber (with $\sigma \cong 46$ deg.) Figure 4 shows a typical comparison of moments and Figure 5 the distortion of the emittance ellipse. This comparison provides a check on both the theory and the simulation work, and lends credence to the simulation of distributions which are not accessible to theory.

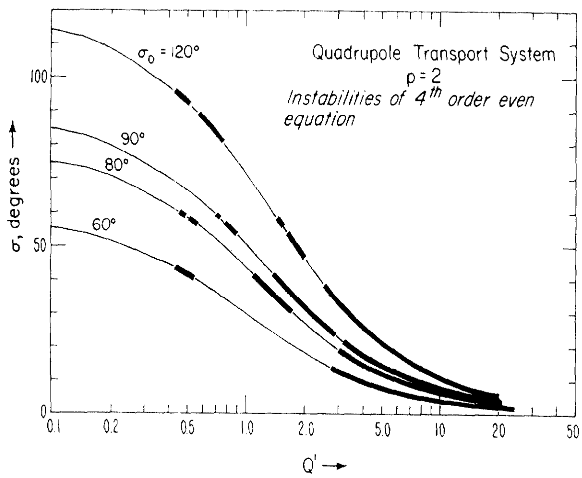

Fig. 1. Regions of fourth-order even instability (heavy lines) for a symmetric FODO quadrupole lattice with a magnet occupancy factor $\eta = 1/2$.

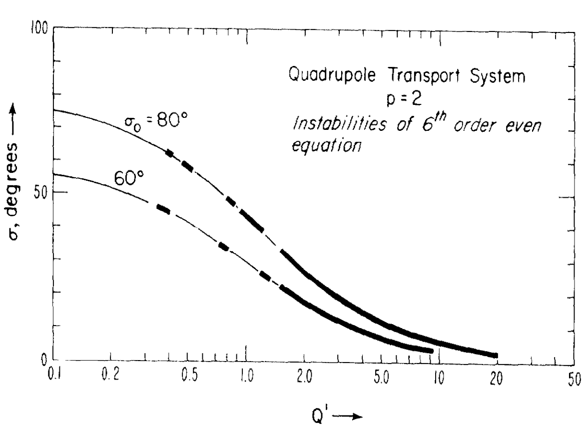

Fig. 2. Regions of sixth-order even instability for $\eta = 1/2$.

Conclusions

Because of the singular nature of the K-V distribution, it is somewhat more susceptible to instability than a more realistic distribution. Therefore, avoidance of these instabilities should provide a conservative criterion for design of a transport channel. In this spirit, the zero-intensity phase advance should be less than 60° in order to avoid the envelope and third order instabilities and then one should limit the current to a tune depression of a factor of 2.5 (e.g., 60° to 24°), at which intensity the extended unstable range begins to appear. In the notation of eq'n (3) of reference (1), the corresponding figure of merit, $Q'/u_m^{2/3}$, is then only a function of the fraction of the channel occupied by quadrupoles, as shown in Table I.

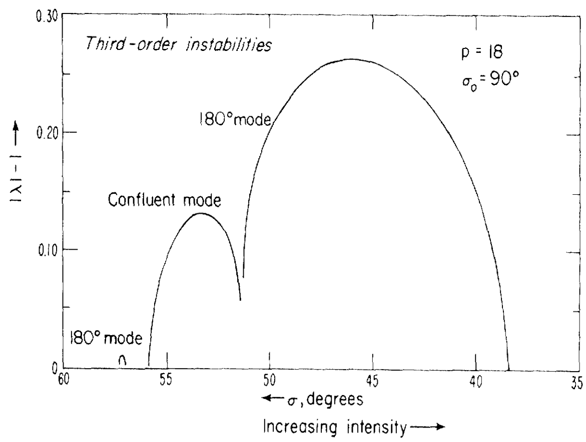

Fig. 3. Instability regions for the third-order mode. The curves represent the growth per period calculated for $\eta = 1/10$, but are very insensitive to η when plotted vs. σ.

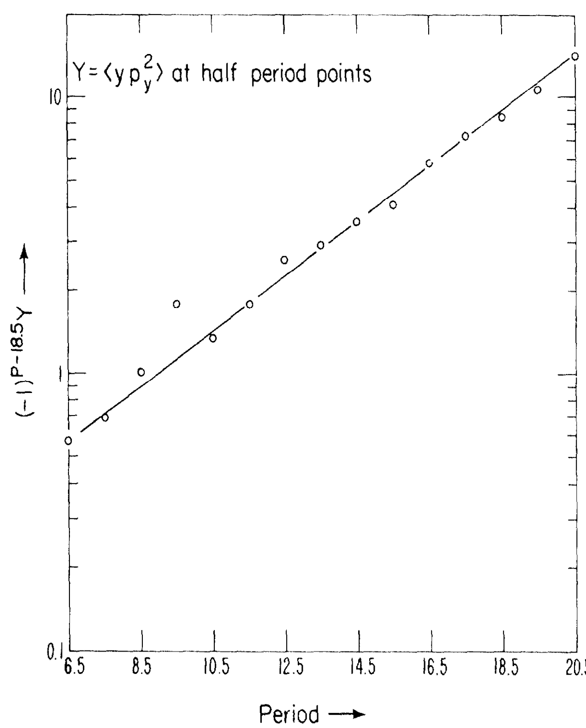

Fig. 4. Illustration of exponential growth of $| < y\, p_y^2 > |$ found in simulation computations and attributed to the third-order mode instability.

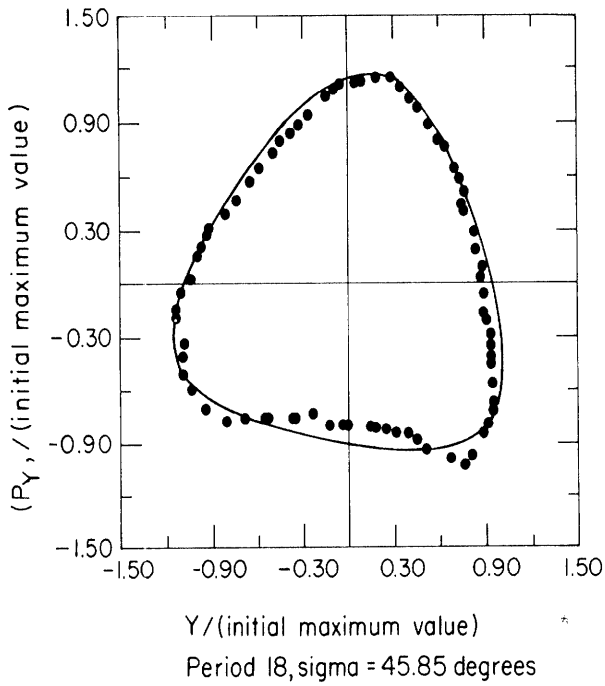

Fig. 5. Boundary of computed distribution in a y, p_y projection.

Acknowledgments

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Table I.

Figures of Merit for σ_0 = 60 Deg. & σ = 24 Deg.

η*	$Q' = Q/(\varepsilon\sqrt{K})$	$[FM] = Q'/u_m^{2/3}$
1	1.6581	0.764
2/3	1.5392	0.688
1/2	1.3959	0.601
1/3	1.1851	0.481
1/4	1.0445	0.405
1/5	0.9436	0.354
1/6	0.8669	0.315
1/8	0.7567	0.263
1/10	0.6799	0.228

*η denotes the magnet occupancy factor.

References

1. G. Lambertson, L.J. Laslett, and L. Smith, IEEE Trans., Nucl. Sci., NS-24, p. 993, June 1977.

2. R.L. Gluckstern, Proc. 1970 Proton Linac Conference (NAL), p. 811.

3. The theory and results are described in detail in LBL HI-FAN notes 13, 14, 15, 43 and 44.

4. See I. Haber, report in proceedings of this conference.