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LONGITUDINAL MOTION IN HIGH CURRENT ION BEAMS - A SELF-CONSISTENT PHASE SPACE DISTRIBUTION WITH AN ENVELOPE EQUATION*

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Summary

Many applications of particle acceleration, such as heavy ion fusion, require longitudinal bunching of a high intensity particle beam to extremely high particle currents with correspondingly high space charge forces. This requires a precise analysis of longitudinal motion including stability analysis. Previous papers[1,2] have treated the longitudinal space charge force as strictly linear, and have not been self-consistent; that is, they have not displayed a phase space distribution consistent with this linear force so that the transport of the phase space distribution could be followed, and departures from linearity could be analyzed. This is unlike the situation for transverse phase space where the Kapchinskij-Vladimirskij (K-V)[3] distribution can be used as the basis of an analysis of transverse motion. In this paper we derive a self-consistent particle distribution in longitudinal phase space which is a solution of the Vlasov equation and derive an envelope equation for this solution. The solution is developed in Section II from a stationary solution of the Vlasov equation derived in Section I.

I. An Example of a Stationary Distribution for Longitudinal Transport

In these calculations we assume the longitudinal and transverse motion of particles in the beam bunch are completely decoupled with the beam length much greater than the beam radius. We choose the longitudinal distance from the center of the bunch z and the position of the center of the bunch s as the dependent and independent variables.

The ions in the bunch experience a space charge force given by [4]

$$F_z = \frac{-g}{\gamma^2} q^2 e^2 \frac{d\lambda}{dz} \qquad (1)$$

where q is the ion charge state, λ is the number of ions per unit length, and g is a geometrical factor of order unity.

We simplify the discussion by assuming the center of the bunch is not accelerating but moves with constant speed βc, and rewrite (1) (non-relativistically) as

$$\frac{d^2 z}{ds^2} = z'' = -\frac{q^2 e^2 g}{\beta^2 c^2 M} \frac{d\lambda}{dz} \equiv - A \frac{d\lambda}{dz} \qquad (2)$$

where M is the ion mass and the ' symbol denotes differentiation with respect to s. This is a debunching force and tends to extend the bunch. We add a linear bunching force F_B by applying a linearly ramped external electric field $E_z = E'z$ so that $F_B = qE'z$. We define a bunching parameter K by the equation $F_B = - MK\beta^2 c^2 z$ and obtain the equation of motion

$$z'' = - A \frac{d\lambda}{dz} - Kz \qquad (3)$$

and we associate this equation of motion (3) with a massless Hamiltonian:

$$H \equiv \frac{z'^2}{2} + A\lambda + \frac{Kz^2}{2} + H_0 \qquad (4)$$

The Vlasov equation for the z-z' phase space distribution f(z,z',s) from this Hamiltonian is:

$$\frac{\partial f}{\partial s} + z' \frac{\partial f}{\partial z} + (- A \frac{\partial \lambda}{\partial z} - Kz) \frac{\partial f}{\partial z'} = 0 \qquad (5)$$

If we choose f such that f = f(H) then the Vlasov equation requires that $\frac{\partial f}{\partial s} = 0$ and we have a stationary distribution. We must also choose an f that is self-consistent; that is

$$\int f(z', z, s)\, dz' = \lambda(z, s) \qquad (6)$$

As a simplest solution we desire $\lambda(z)$ to be parabolic:

$$\lambda = \frac{3}{4} \frac{N}{z_m} (1 - \frac{z^2}{z_m^2}) \equiv \lambda_0 (1 - \frac{z^2}{z_m^2}) \qquad |z| < z_m$$

$$\lambda = 0 \qquad |z| > z_m \qquad (7)$$

where N is the total number of ions in the bunch and $2 z_m$ is the length of the bunch. This gives a linear space charge force of $\frac{2A\lambda_0 z}{z_m^2}$. We choose $H_0 = - A\lambda_0$ and define a reduced bunching parameter K^1 by

$$H = \frac{z'^2}{2} + \frac{1}{2} (K - \frac{2A\lambda_0}{z_m^2}) z^2 = \frac{z'^2}{2} + \frac{K^1}{2} z^2 .$$

$$\text{with } K^1 \equiv K - \frac{2A\lambda_0}{z_m^2} \qquad (8)$$

For an ansatz we choose:

$$f(H) = C\sqrt{2(H_{max} - H)} = C\sqrt{2H_{max} - K^1 z^2 - z'^2} \quad \text{for } 0 < H < H_{max}$$

$$f(H) = 0 \text{ for } H > H_{max} \qquad (9)$$

Writing $2 H_{max} \equiv K^1 z_m^2$, we check that this distribution satisfies the condition (6), that is

$$\int_{z'_{min}}^{z'_{max}} C\sqrt{K^1 z_m^2 - K^1 z^2 - z'^2}\, dz' = \lambda(z) \qquad (10)$$

The limits of integration can be found from (9):

$$z'_{max} = -z'_{min} = \sqrt{K^1 (z_m^2 - z^2)}$$

We integrate and find:

$$\lambda(z) = \frac{C\pi K^1 z_m^2}{2} (1 - \frac{z^2}{z_m^2}) \qquad (11)$$

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This agrees with (7) if we choose the value of C as

$$C = \frac{2\lambda_0}{\pi K^1 z_m^2} = \frac{2\lambda_0}{\pi(Kz_m^2 - 2A\lambda_0)} \quad (12)$$

The three interdependent parameters λ_0, z_m, K define a consistent solution. Two of the parameters are set by internal properties of the beam bunch: its longitudinal emittance ε_L and the total number of ions N. We display the relationships:

$$N = \frac{4}{3}\lambda_0 z_m \quad (13)$$

and

$$\varepsilon_L = z_{max} \cdot z'_{max}\ (z = 0) = (K^1)^{1/2} z_m^2$$

$$= (K - \frac{2A\lambda_0}{z_m^2})^{\frac{1}{2}} z_m^2 \quad (14)$$

The third parameter, which we associate with the external field K, may be chosen arbitrarily, but K must be positive. Equations (13, 14) can be combined to give

$$\varepsilon_L^2 + \frac{3}{2} ANz_m - Kz_m^4 = 0 \quad (14A)$$

which can then be solved for z_m.

Consideration of the three dimensional problem with the requirement of transverse (x-y) stability may set an upper limit λ_{max} on λ_0, the maximum ion density. This would set a lower limit on $z_m (z_m > \frac{3}{4}\frac{N}{\lambda_{max}})$, which would mean an upper limit on K obtainable from the solution of equation (14A) with the limiting value of z_m.

II. Envelope Equation for Longitudinal Motion

The existence of a stationary solution of the longitudinal transport problem of form (9) suggests that a similar solution f(z, z', s) that is not stationary can be found. As our ansatz we choose

$$f(z,z',s) = D\sqrt{1 - \frac{z^2}{z_0^2} - \frac{(z' - Bz)^2}{a_s^2}} \quad (15)$$

in the region $\frac{z^2}{z_0^2} + \frac{(z' - Bz)^2}{a_s^2} < 1$

(f(z, z', s) = 0 otherwise),

where the parameters D, z_0, a_s, B are functions of s whose behavior is determined by requiring that f satisfy the Vlasov equation.

We find:

$$\lambda = \int_{z'_{min}}^{z'_{max}} D\sqrt{1 - \frac{z^2}{z_0^2} - \frac{(z' - Bz)^2}{a_s^2}}\, dz'$$

$$= \frac{\pi}{2} a_s D(1 - \frac{z^2}{z_0^2}) \equiv \lambda_0 (1 - \frac{z^2}{z_0^2}) \quad (16)$$

We include an external linear focussing field of arbitrary value Kz, so that

$$z'' = -A\frac{d\lambda}{dz} - Kz = (\frac{2A\lambda_0}{z_0^2} - K)\, z \quad (17)$$

From the Vlasov equation:

$$\frac{\partial f}{\partial s} + z'\frac{\partial f}{\partial z} + z''\frac{\partial f}{\partial z'} = 0 \quad (18)$$

We find $\frac{\partial D}{\partial s} = 0$ (18A)

$$a_s = \frac{\varepsilon_L}{z_0}, \text{ where } \varepsilon_L \text{ is some constant} \quad (18B)$$

$$B = \frac{dz_0}{ds} / z_0 = \frac{z'_0}{z_0} \quad (18C)$$

and we find an envelope equation for z_0

$$\frac{d^2 z_0}{ds^2} = \frac{\varepsilon_L^2}{z_0^3} + \frac{2A\lambda_0}{z_0} - Kz_0 \quad (18D)$$

Identifying λ_0 as $\frac{3}{4}\frac{N}{z_0}$ from (7)

and noting $D = \frac{3N}{2\pi\varepsilon_L}$ from (16),

we obtain a self-consistent solution to the Vlasov equation:

$$f(z,z',s) = \frac{3N}{2\pi\varepsilon_L}\sqrt{1 - \frac{z^2}{z_0^2} - \frac{z_0^2}{\varepsilon_L^2}(z' - \frac{z'_0}{z_0} z)^2} \quad (19)$$

wherever the argument of the square root is real (f = 0 otherwise), and where z_0 is a solution of the envelope equation:

$$\frac{d^2 z_0}{ds^2} = \frac{\varepsilon_L^2}{z_0^3} + \frac{3}{2}\frac{AN}{z_0^2} - Kz_0 \quad (20)$$

The initial conditions $z_0(s = 0)$ and $z'_0\ (s = 0)$ may be chosen arbitrarily and K may be an arbitrary function of s.

This equation is the same envelope equation derived earlier from a non-self-consistent perspective[1] and is similar to the K-V equations. The quantity $\frac{z^2}{z_0^2} + \frac{z_0^2}{\varepsilon_L^2}(z' - \frac{z'_0}{z_0} z)^2$ is an invariant similar to the transverse Courant-Snyder invariant.[5] The stationary solution derived in the previous section is simply the special solution of equations (19, 20) where ε_L, z_0, K and N are chosen in a combination such that $\frac{d^2 z_0}{ds^2}$ is set equal to zero, and also z'_0 is initially zero.

The similarity between the longitudinal envelope equation (20), and the K-V equations suggests that an analysis of stability of the longitudinal envelope similar to the earlier analysis of the transverse K-V envelope[6] would be of similar value in identifying instabilities. Results of this analysis will be presented in future papers.

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