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POTENTIAL AND FIELD PRODUCED BY A UNIFORM OR NON-UNIFORM ELLIPTICAL BEAM INSIDE A CONFOCAL ELLIPTIC VACUUM CHAMBER

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1. INTRODUCTION

The potential produced by an isolated beam of elliptic cross-section seems to have been considered first by L.C. TENG [1]. Image effects of line charges in elliptic vacuum chambers were introduced into accelerator theory by L.J. LASLETT [2]. Various approximate solutions for elliptic beams of finite cross-section coasting inside an elliptic vacuum chamber were subsequently proposed by P. LAPOSTOLLE [3] and C. BOVET [4].

In this paper a rigourous expression will be derived for the potential produced by an elliptic beam inside an elliptic vacuum chamber , provided the beam envelope and the vacuum chamber can be assimilated to confocal ellipses.

2. THE UNIFORM BEAM

Let ρ_o be the charge density, 2g and 2p the axes of the ellipse representing the beam envelope, 2G and 2P the axes of the ellipse representing the vacuum chamber, 2c the distance between foci (Fig 1) and Φ the potential. The nature of the problem suggests using elliptic coordinates. We put

$$\begin{aligned} x &= c\,\mathrm{ch}\eta\,\cos\Psi , && 0 \leqslant \eta < \infty \\ y &= c\,\mathrm{sh}\eta\,\sin\Psi && -\pi < \Psi \leqslant \pi \end{aligned} \tag{1}$$

If we define the vacuum chamber by η_1 and the beam envelope by η_2 we have

$$\begin{aligned} G &= c\,\mathrm{ch}\eta_1 , && P = c\,\mathrm{sh}\eta_1 \\ g &= c\,\mathrm{ch}\eta_2 , && p = c\,\mathrm{sh}\eta_2 \end{aligned} \tag{2}$$

and the confocality condition can be written

$$G^2 - P^2 = g^2 - p^2 = c^2 \tag{3}$$

Symmetry of the potential imposes the constraints

$$\Phi(\eta,-\Psi) = \Phi(\eta,\Psi) , \qquad \Phi(\eta,\pi-\Psi) = \Phi(\eta,\Psi) \tag{4}$$

while symmetry of the electric field requires

$$\left(\frac{\partial\Phi}{\partial\eta}\right)_{\eta=o} = 0 , \quad \left(\frac{\partial\Phi}{\partial\Psi}\right)_{\Psi=o} = 0 , \quad \left(\frac{\partial\Phi}{\partial\Psi}\right)_{\Psi=\frac{\pi}{2}} = 0 \tag{5}$$

In elliptic coordinates Laplace's equation is simply

$$\frac{\partial^2\Phi}{\partial\eta^2} + \frac{\partial^2\Phi}{\partial\Psi^2} = 0 \tag{6}$$

and the solutions are of the form

$$\Phi = A\eta + B , \qquad \Phi = C(\eta^2 - \Psi^2) + D\Psi \tag{7}$$

$$\Phi = (a\,\mathrm{ch}n\eta + b\,\mathrm{sh}n\eta)(c\cos n\Psi + d\sin n\Psi)$$

Taking into account the periodicity relation $\Phi(\eta,\Psi+2\pi) = \Phi(\eta,\Psi)$ and the symmetry requirements (4) and (5), the general solution of Laplace's equation can be written

$$\Phi_L = \sum_{n=1}^{\infty} (a_n \mathrm{ch}2n\eta + b_n \mathrm{sh}2n\eta)\cos 2n\Psi + A\eta + B \tag{8}$$

with n integer.

To solve Poisson's equation

$$\frac{\partial^2\Phi}{\partial\eta^2} + \frac{\partial^2\Phi}{\partial\Psi^2} = -\frac{\rho c^2}{\varepsilon_o}(\mathrm{ch}^2\eta - \cos^2\Psi) \tag{9}$$

in the case of constant density we put

$\Phi = \Omega - \dfrac{\rho_o c^2}{8\varepsilon_o}(\mathrm{ch}2\eta + \cos 2\Psi)$ and are thus reverted to a Laplace equation in Ω. The solution of Poisson's equation can therefore be written

$$\Phi_P = \sum_{n=1}^{\infty}(\alpha_n \mathrm{ch}2n\eta + \beta_n \mathrm{sh}2n\eta)\cos 2n\Psi + E\eta + F - \frac{\rho_o c^2}{8\varepsilon_o}(\mathrm{ch}2\eta + \cos 2\Psi) \tag{10}$$

Assuming now that the vacuum chamber is at zero potential and writing the continuity conditions for the potential and the field at the transition between the Laplace region and the Poisson region, all coefficients in Eqs (8) and (10) can be determined. Putting $2\eta = \xi$, $2\Psi = \phi$, the result can be written [5]

$$\Phi_L = \frac{\rho_o c^2}{8\varepsilon_o}\left[\xi_1 - \xi - \frac{\mathrm{sh}(\xi_1 - \xi)}{\mathrm{ch}\xi_1}\cos\phi\right]\mathrm{sh}\xi_2 \tag{11}$$

$$\Phi_P = \Phi_L + \frac{\rho_o c^2}{8\varepsilon_o}\left\{[\mathrm{ch}(\xi-\xi_2)-1]\cos\phi + \mathrm{ch}\xi_2 - \mathrm{ch}\xi + (\xi-\xi_2)\mathrm{sh}\xi_2\right\} \tag{12}$$

In elliptic coordinates the Laplace potential is simpler than the Poisson potential ; in cartesian coordinates however the reverse is true. Putting $z^4 = (x^2-y^2-c^2)^2 + 4x^2y^2$ the above expressions can indeed be written in the form

$$\Phi_P = \frac{\rho_0 g p}{2\varepsilon_0}\left[\frac{GP}{G^2+P^2} + \ln\frac{G+P}{g+p} - \left(2\frac{GP}{G^2+P^2} - \frac{p}{g}\right)\left(\frac{x}{c}\right)^2 - \left(\frac{g}{p} - 2\frac{GP}{G^2+P^2}\right)\left(\frac{y}{c}\right)^2 \right] \tag{13}$$

$$\Phi_L = \frac{\rho_0 g p}{2\varepsilon_0}\left[\frac{GP}{G^2+P^2}\left(1 - 2\frac{x^2-y^2}{c^2}\right) + \ln\frac{(G+P)\sqrt{2}}{\sqrt{x^2+y^2+z^2+c^2} + \sqrt{x^2+y^2+z^2-c^2}} \right.$$
$$\left. + \frac{1}{c^2\sqrt{2}}\left(|x|\sqrt{x^2-y^2+z^2-c^2} - |y|\sqrt{y^2-x^2+z^2+c^2}\right) \right] \tag{14}$$

The potential inside the beam being parabolic in x and y, we are led to linear expressions for the field components, viz.

$$E_x = \frac{\rho_0 g p}{\varepsilon_0 c^2}\left(2\frac{GP}{G^2+P^2} - \frac{p}{g}\right)x \tag{15}$$

$$E_y = \frac{\rho_0 g p}{\varepsilon_0 c^2}\left(\frac{g}{p} - 2\frac{GP}{G^2+P^2}\right)y \tag{16}$$

The field components outside the beam are more complicated. One finds

$$|E_x| = \frac{\rho_0 g p}{\varepsilon_0 c^2}\left(2\frac{GP}{G^2+P^2}|x| - \sqrt{\frac{x^2-y^2+z^2-c^2}{2}}\right) \tag{17}$$

$$|E_y| = \frac{\rho_0 g p}{\varepsilon_0 c^2}\left(\sqrt{\frac{y^2-x^2+z^2+c^2}{2}} - 2\frac{GP}{G^2+P^2}|y|\right) \tag{18}$$

The potential has a maximum at the centre of the beam. From Eq (13) we find

$$\Phi_{max} = \frac{\rho_0 g p}{2\varepsilon_0}\left(\frac{GP}{G^2+P^2} + \ln\frac{G+P}{g+p}\right) \tag{19}$$

Letting $G \to \infty$, $P \to \infty$ in Eqs (15-18) we find for the components of the electric field inside an isolated beam

$$E_{ix} = \frac{\rho_0 p x}{\varepsilon_0 (g+p)} \tag{20}$$

$$E_{iy} = \frac{\rho_0 g y}{\varepsilon_0 (g+p)} \tag{21}$$

and for the same components outside the isolated beam

$$|E_{ix}| = \frac{\rho_0 g p}{\varepsilon_0 c^2}\left(|x| - \sqrt{\frac{x^2-y^2+z^2-c^2}{2}}\right) \tag{22}$$

$$|E_{iy}| = \frac{\rho_0 g p}{\varepsilon_0 c^2}\left(\sqrt{\frac{y^2-x^2+z^2+c^2}{2}} - |y|\right) \tag{23}$$

The effect of the vacuum chamber wall is obtained by taking the difference between the field vectors in the presence and in the absence of the vacuum chamber ; this is usually referred to as an "image effect". We find for the components of this field

$$E_{vc\,x} = -\frac{\rho_0 g p}{\varepsilon_0 c^2}\frac{(G-P)^2}{G^2+P^2}x \tag{24}$$

$$E_{vc\,y} = +\frac{\rho_0 g p}{\varepsilon_0 c^2}\frac{(G-P)^2}{G^2+P^2}y \tag{25}$$

The field due to the presence of the vacuum chamber is everywhere linear, it reduces the field due to the isolated beam in the direction parallel to the major axis and increases the isolated beam field in the direction parallel to the minor axis.

3. THE NON-UNIFORM BEAM

We now allow the density distribution to be η-dependent and make two assumptions :

a) With the exception of a scaling factor (which will depend on the density distribution), we assume that in the Laplace region, the potential is the same as in the case of a uniform beam.

b) We assume that, as in the case of a uniform beam, only the second harmonic of ψ will survive in the Fourier expansion of the potential distribution.

If we can solve Laplace's and Poisson's equations on the basis of these assumptions and if we can make these solutions obey all boundary conditions, we shall have solved the potential problem by virtue of the uniqueness theorem.

According to the first assumption we write for the Laplace potential

$$\Phi_L = \frac{c^2}{2\varepsilon_0}K\left[\eta_1 - \eta - \frac{\mathrm{sh}2(\eta_1-\eta)}{2\mathrm{ch}2\eta_1}\cos 2\psi\right] \tag{26}$$

where only K is assumed to depend on $\rho(\eta)$. According to the second assumption we put

$$\Phi_P = G(\eta) + H(\eta)\cos 2\psi \tag{27}$$

Replacing these values in Eq (9) we can separate Poisson's equation into

$$H'' - 4H = \frac{\rho(\eta)c^2}{2\varepsilon_0} \tag{28}$$

and

$$G'' = -\frac{\rho(\eta)c^2}{2\varepsilon_0}\ \mathrm{ch}2\eta \tag{29}$$

Solving these equations and taking into account the boundary conditions one finds for the potential [6]

$$\phi = \frac{c^2}{8\varepsilon_0}\left\{C(\xi_2)\left[\ \xi_1-\xi_2-\frac{\mathrm{sh}(\xi_1-\xi)}{\mathrm{ch}\xi_1}\cos\phi\ \right]+\int_\xi^{\xi_2} C(u)du \right. \tag{30}$$
$$\left. +\ [\overline{S(\xi)}\mathrm{ch}\xi - \overline{C(\xi)}\mathrm{sh}\xi]\ \cos\phi\right\}$$

where $\xi = 2\eta$, $\phi = 2\Psi$ and we have put for abbreviation

$$C(\xi) = \int_0^\xi \rho(\tfrac{s}{2})\ \mathrm{ch}s ds \quad , \qquad S(\xi) = \int_0^\xi \rho(\tfrac{s}{2})\ \mathrm{sh}s ds \tag{31}$$

$$\overline{C(\xi)} = C(\xi_2) - C(\xi) \qquad \overline{S(\xi)} = S(\xi_2) - S(\xi) \tag{32}$$

Eq (30) applies to the Poisson as well as to the Laplace region. In the latter case $\overline{C(\xi)} = \overline{S(\xi)} = 0$ and therefore

$$\phi_L = \frac{c^2}{8\varepsilon_0}\ C(\xi_2)\left[\ \xi_1 - \xi - \frac{\mathrm{sh}(\xi_1 - \xi)}{\mathrm{ch}\xi_1}\cos\phi\ \right] \tag{33}$$

The field components are given by

$$E_x = \frac{1}{2\varepsilon_0}\ [C(\xi_2)\mathrm{th}\xi_1 - S(\xi_2) + S(\xi) - C(\xi)\mathrm{th}\ \tfrac{\xi}{2}\ ]x \tag{34}$$

$$E_y = \frac{1}{2\varepsilon_0}\ [S(\xi_2) - C(\xi_2)\mathrm{th}\xi_1 - S(\xi) + C(\xi)\mathrm{coth}\ \tfrac{\xi}{2}]y \tag{35}$$

inside as well as outside the beam. The first two terms in these expressions represent the linear part of the field, whereas the last two terms are non-linear contributions. If $\rho = \rho_0$ = const we have $S(\xi) - C(\xi)\mathrm{th}\ \frac{\xi}{2} = C(\xi)\mathrm{coth}\ \frac{\xi}{2} - S(\xi) = 0$ and the field is strictly linear inside the beam.

For the image fields one finds [6]

$$E_{vc\ x} = -\frac{C(\xi_2)}{2\varepsilon_0}\ \frac{(G - P)^2}{G^2 + P^2}\ x$$

$$E_{vc\ y} = +\frac{C(\xi_2)}{2\varepsilon_0}\ \frac{(G - P)^2}{G^2 + P^2}\ y$$

The "reflection" of an elliptic beam in an elliptic vacuum chamber produces two perpendicular virtual beams of opposite charge.

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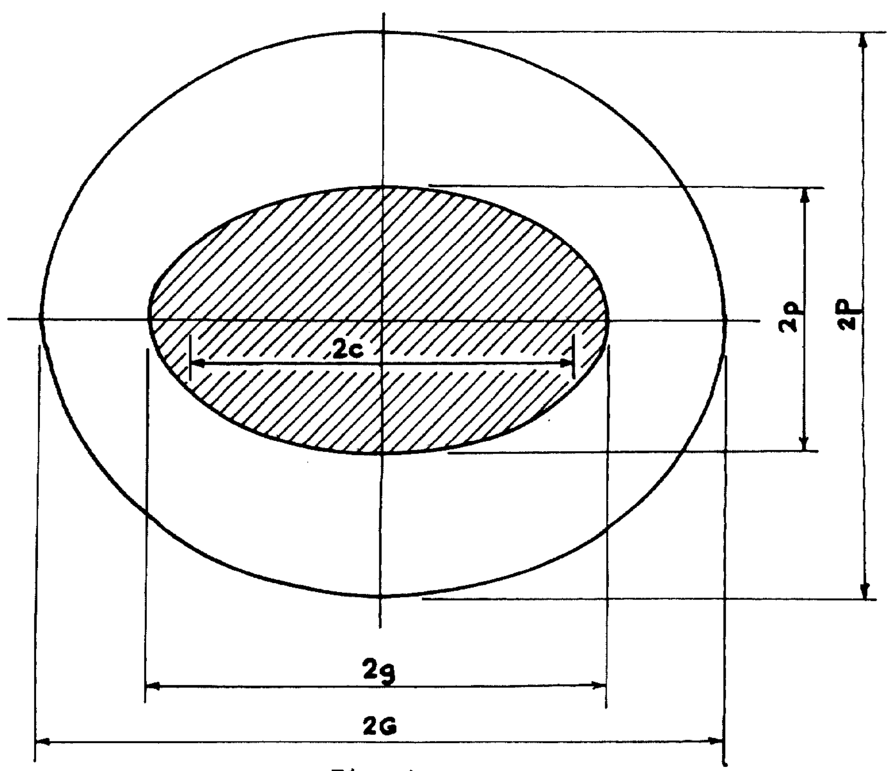

Fig. 1