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# A FOOTBALL COIL, A DEVICE TO PRODUCE ABSOLUTE MINIMUM MAGNETIC FIELD AND AN ISOCHRONOUS CYCLOTRON FOR HEAVY IONS * +

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## Introduction

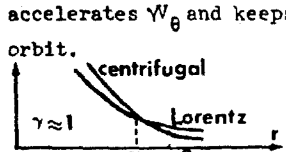
The electric solenoid considered here consists of several discrete, circular and superconducting wires. The size of each loop varies from one to several meters in the radius. Furthermore, if such a solenoid is made into a football shape by squeezing the ends symmetrically, it is referred to here as a football coil. When a constant external current $\underset{\sim}{J}^E$ flows through the football coil, the net magnetic field $\underset{\sim}{B}^E(r,z)$ produced at the symmetry center (r=z=0) is parallel to the z-axis and is independent of the azimuthal angle $\theta$ in the cylindrical coordinates. It turns out that $B^E$ is the absolute minimum at the center. To understand this, let us consider the simplest football coil consisting of three current loops. The end loops located at $z = \pm 1$ have the identical radius $r^M$, while the central loop located at z = 0 has a large radius $r^C$. The end loops alone produce the well-known mirror field $\underset{\sim}{B}^M(r, z)$, which is not the absolute minimum at the center. In fact, the center is a saddlepoint, namely an axially minimal $B^M(0, z)$ on a radially maximal $B^M(r, 0)$. Therefore, the large central loop is introduced in Fig(4) producing an extra field $\underset{\sim}{B}^C(r, z)$, which increases in the central plane toward its current carrying wire according to the Amperé law. Since the $\underset{\sim}{B}^C(r, 0)$ dominates over the radially decreased mirror field $\underset{\sim}{B}^C(r, 0)$, the net field $\underset{\sim}{B}^E = \underset{\sim}{B}^M + \underset{\sim}{B}^C$ becomes the absolute minimum at the mirror field saddlepoint. In general, for a multiple turn football coil $r^C > r^M$'s, the minimal field spot at $z \approx 0$ and $r^0 \approx 0$ is useful for a plasma confinement experiment. If the central loop now has a small radius $r^C < r^M$'s, such a football coil is so to speak deflated in the middle. Since the magnetic field line wraps around its electric wire, the direction of $\underset{\sim}{B}^C$ outside the small central wire anti-parallel to those inside is therefore reversed from strong $\underset{\sim}{B}^M$ outside the small central wire. In Fig(4b), the magnitude of $\underset{\sim}{B}^C$ decreases outside its wire, the net $\underset{\sim}{B}^E$ outside the central wire becomes the minimum over a circle of radius $r^0$ where $r^C < r^0 < r^M$'s. Such a coil (20) never decays in contrast to Christofilos' tandem. When the deflated football coil is applied to a particle accelerator, an important effect referred to here as the synergic focusing seems to be overlooked in the present technological viewpoint in, e.g., the biannual Particle Accelerator Conferences at USA since 1969.

## A New Heavy Ion Cyclotron

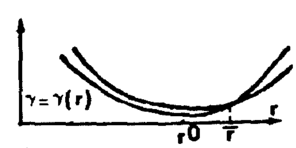
The synergic focusing provided by the coil combines the concept[1] of weak focusing with the Fermi idea of magnetic mirroring. The Lorentz force $\rho_q \underset{\sim}{V} \times \underset{\sim}{B}^E = (F_r^E, F_\theta^E, F_z^E)$ focuses ions into the orbit $(\bar{r}, \bar{\theta}, \bar{z})$, while the boosting field $E^P$ produced by a power supply accelerates ions. There are two stages of acceleration. At the first stage, the ions injected at $\bar{r}^{IN}$ $(r^C < \bar{r}^{IN} < r^0)$ with the velocity $V_\theta^{IN}$, that are boosted, spiralled by $F_\theta^P$, $F_r$ respectively, are focused both axially and radially by the slowly decreased $B_z^E(r, 0) \sim \bar{r}^{-(\epsilon-1)}$ $1 > \epsilon > 0$, as used in an old cyclotron. After having gained MeV kinetic energy $T \equiv \mathcal{E} - m_o c^2$ with the positive time rate

$$\dot{T} = \dot{\mathcal{E}} = m_o c^2 \dot{\gamma} = q(\underset{\sim}{V} \cdot \underset{\sim}{E}^P) > 0, \qquad (1)$$

the ions arrive to the minimal $B_z(r^0, 0)$ at the radius $r^0$ at the end of the first stage. At the second stage which goes beyond the conventional cyclotron, the relativistic mass $m_o\gamma$ increased appreciably by the further acceleration must be matched by a radially increased $B_z(r > r^0, 0)$. The purpose is to keep the ion cyclotron frequency $\Omega(r) \equiv (q B_z / m_o \gamma c)$ $\equiv$ constant synchronized for all r. Then a fixed-duty-cycle power supply,[2] for example the magnetron placed at the center of the deflated football coil and attached radially with long horns for the purpose of resonant cavities, can accelerate MeV ions to GeV ions continuously in a phase-stable region. To intensify $J^I$ one[3] can apply an extra axial current $J_z^E$ that produces $B_\theta^E$ and stores milli-Amp's into a ring at the end of each stage up to multi-Amp's intensity of ion current. Conventional cyclotron radii, for example 5 meters for 1 GeV $^4He^{++}$ ion in 10 K Gauss cyclotron field, are reduced together with the vacuum chamber, etc. Because the insufficient cyclotron field produced by an iron magnet is technologically augmented by the available superconducted football coil (20). When $B_z^E$ increases radially, orbital radius $\bar{r} = m_o \gamma |V_\theta| c / qB_z^E = [T(T + 2m_o c^2)]^{\frac{1}{2}} / qB_z^E \approx T/qB_z^E$ is relatively reduced since T also increases. When $\dot{\gamma} \approx 0$, $\bar{r}B_z^E \approx T/q \approx$ constant if and only if $\bar{r} \approx T/B_z^E q \approx$ constant. The former is the fact used[1] in an old cyclotron to weakly focus the deviant ions by a radially decreased $B_z^E$, see Fig. (1). The latter is the fact used in the present two-stage cyclotron, see Fig. (2). To reinforce the radial focusing, it turns out that the $E_\theta^P(r)$ is radially increased, which accelerates $\gamma W_\theta$ and keeps the deviant ions in their orbit.




fig(1)




fig(2)

+Navy Case Number 61,896 on 11-th April 1977

*dedicated to Professor George E. Uhlenbeck who once said that a research like an art is a self expression, which happens to be useful.

## Theory of Synergic Focusing

To define it rigorously, an ion at $\tilde{x} \equiv (\bar{r}, r\bar{\theta}, \bar{z})$ is under the force of the total electromagnetic field, $E = E^E + E^I$, $B = B^E + B^I$, produced by external current $J^E(x,t)$ and by accelerated ion fluid $J^I(x,t) = qn(x,t)V(x,t)$. Therefore the Gauss and Biot-Savart laws are generalized here to include arbitrary time dependence. Solving Maxwell's equations $\Box^2(\Phi, A) = -4\pi(\rho_q, c^{-1}J)$ of electromagnetic potentials $(\Phi, A)$, one uses Green's function $\mathcal{G} = \delta(t' + c^{-1}|x-x'| - t)|x-x'|^{-1} + \mathcal{G}^H$ satisfying boundary conditions in obtaining $E \equiv -\nabla\Phi - c^{-1}\partial_t A$ and $B \equiv \nabla \times A$, including the radiation loss $cE \times B/4\pi$.

$$E = E^H + \int[(1+c^{-1}d_{t'}|R|)\rho_q' R - c^{-1}d_{t'}|R|^2 J'] |R|^{-3} dx' \qquad (2a)$$

$$cB = cB^H + \int(1+c^{-1}d_{t'}|R|)J' \times R|R|^{-3}dx' \qquad (2b)$$

where the vector $R = x - x'$ points to field point $x$ from sources $\rho_q' = qn(x',t')$ and $J' = J(x',t')$ and the total time derivative $d_{t'} \equiv (d/dt')$ evaluates at a retarded time $t' = t - c^{-1}|R|$. From these general appearances follow that the linear superposition principle used in the introduction and the time average later is valid. Of course, the time-independent $J = J^E(x) = J^E_\theta(r,z)\hat{\theta} \equiv J^E_\theta$ reduces (2b) to the Biot-Savarat law giving $J^E_\theta \times R = (J^E_\theta R_z;\ 0;\ -J^E_\theta R_r)$ respectively

$$(B^E_r = 0,\ z = 0;\ B^E_\theta \equiv 0;\ B^E_z > 0,\ J^E_\theta > 0) \qquad (3a)$$

for the $x$ lying mainly inside the coil $x'$, $R_r = x_r - x_r' < 0$. In order to push the beam inward and the coil outward $\int J^I(x,t) \times B^E(x)dxc^{-1} = -\int J^E(x) \times B^I(x,t)dxc^{-1}$, the beam-coil interaction force is therefore repellant. Accordingly, the positive ion beam must be accelerated in the opposite direction to $J^E$,

$$J^I \approx J^I_\theta\hat{\theta},\ V_\theta < 0;\ E^P \approx E^P_\theta\hat{\theta}, E^P_\theta < 0;\ B^I_z < -B^E_z. \qquad (3b)$$

Averaging by tracing the field point of (2b) on the beam boundary gives a longitudinal beam dimension $|R_\theta| \gg |R_r| > |R_z|$ that implies for the ion selffield a reversed inequality

$$|\overline{B^I_z}| > |\overline{B^I_r}| \gg |\overline{B^I_\theta}| \approx 0. \qquad (3c)$$

Because both $J^{I'}$ and its derivative $d_{t'}J^{I'} \approx \dot{J}_\theta'\hat{\theta}' + J_\theta'\hat{r}'$ during the acceleration $\dot{J}_\theta'|R_\theta|c^{-1} > J_\theta'$ are essentially longitudinal, then their cross products with $R \approx R_\theta$ in (2b) reduce $|\overline{B^I_\theta}|$ to the smallest of all. Since the strong mirror field $B^E$ focuses ions on the median orbital plane with the increasingly favorable criterion $|V_z/V_\theta| < |B^E_z(\pm1) - B^E_z(0))/B^E_z(0)|$ as $|V_\theta|$ being accelerated, then an increasingly better beam geometry $|R_\theta| \gg |R_r| > |R_z|$ implies (2a) on the average a strong inequality among field components $|E^I_\theta| \gg |E^I_r| > |E^I_z|$, namely the maximum of the averaged ion field $\overline{E^I} \approx \overline{E^I_\theta}\hat{\theta}$ having its amplitude $|\overline{E^I_\theta}|$ decayed axially as $|R_z|^{-2}$,

$$z \gtrless 0,\ \partial_z|\overline{E^I_\theta}| \lessgtr 0 \qquad (3d).$$

These time-averaged self fields are typical of the so-called velocity fields since the acceleration fields have been averaged away from (2). Now to justify the time average, the acceleration fields are known to give the radiation loss per unit time, $(c/4\pi)\int E \times Bd^3x \approx (2Nq^2c/3)(\gamma V_\theta/c)^4 r^{-2}$ following the Lienard approximation $d_t(m_o\gamma V_\theta) \approx m_o\gamma V_\theta^2 r^{-1}$. Because the ion inertia is so large that $m_oc^2 = 3.7$ GeV for $He^4$ the loss $\sim (T/m_oc^2)^4$ is negligible. Consequently, these time-averaged self-fields (3) are essential and useful later.

Based on (2,3) a synergic focusing will require two conditions: (i) a radially differential scheme for energizing particles,

$$\partial_r|E^P_\theta| > 0;\ |E^P_\theta(z)| = |E^P_\theta(-z)| \leq |E^P_\theta(0)|;\ E^P_r \equiv 0, \qquad (4a)$$

and (ii) a net Lorentz force for axial focusing,

$$z \gtrless \bar{z} = 0,\ cF_z \equiv \rho_q(V_rB_\theta - V_\theta B_r) \lessgtr 0 \qquad (4b)$$

where $B = B^E + B^I + B^P \equiv (B_r, B_\theta, B_z)$. To satisfy (i), we recapitulate the facts (a, b, c) first. (a) The boosting field $E^P_\theta$ comes from the magnetron radiation during the radial acceleration of electrons which are gyrated and precessed along a circle on the orbital plane under the magnetron crossfields $E^P_r \times B^P_z$ at $z = 0$. (b) The radiation is guided into several long boxes of the size $2\Delta r \gg 2\Delta z > r\Delta\theta$, which have been radially attached to the centrally located magnetron. (c) Slits are cut along both sides of the box from $r = r^{In}$ to $r^{Out} \approx \Delta r$ and the width of the slits must be less than the width of the box $2\Delta z$, which intersects the median plane at $z = 0$. Next, based on (a,b,c), the resonant radiation inside a box consists of a transversal electric wave $(B^P_r, E^P_\theta, B^P_z)$, which requires the longitudinal $E^P_r$ to vanish everywhere and the mixed boundary conditions: $B^P_r(r=0, 2\Delta r) = 0$, $\partial_z B^P_r(z=\pm\Delta z) = 0$. These requirements are satisfied by the $TE_{101}$ wave having the lowest cut-off frequency $\sim (C/2\Delta z\sqrt{\mu\epsilon})$. Let $\xi \equiv \pi r/2\Delta r$, $\zeta \equiv \pi(z+\Delta z)/2\Delta z$. Then a straightforward calculation gives

$$(B^P_r, E^P_\theta, B^P_z) \approx B_o(\sin\xi\cos\zeta,\ b_o\sin\xi\ \sin\zeta,\ c_o\cos\xi\ \sin\zeta)e^{-i\omega t};\ E^P_r \equiv 0 \qquad (5a)$$

Consequently (5a) has the characteristics (4a) satisfied within the first quarter wavelength $0 < \xi < \pi/2$ over $0 < r^C < r^{In} < r^O < r^{Out} \approx \Delta r$. Due to $b_o \approx (i\omega 2\Delta z/\pi c)$, the $|E^P_\theta|$ can further increase if the width $2\Delta z$ flares out radially. In (5a), the magnetic components at $z = 0$.

$$B_r^P = 0,\ B_\theta^P \equiv 0,\ B_z^P \neq 0, \qquad (5b)$$

are similar to (3a) generated by a circular current. Now for (ii) to be satisfied the magnetic flux is to be bulky in the middle and perpendicular to the median plane. Because the continuous flux having radial projection $B_r$ vanished at $z = 0$ must change sign in passing the zero,

$$z \gtrless 0,\ B_r \gtrless 0 \qquad (6)$$

which implies that for $z \gtrless 0$, since we know and (3b) $V_\theta < 0$. $cF_z \approx \rho_q V_\theta B_r \lessgtr 0$ (5b)(3a,b) $B_\theta = B_\theta^I \approx 0$

Let us show here how conditions (i) and (ii) work together in providing the synergic focusing and later the equilibrium orbit. The dynamically stable orbit is defined generally in terms of a net force density $\mathcal{F} = d_t(n\, m_o \gamma \underset{\sim}{V})$ according to the following inequalities:

$$r \gtrless \bar{r},\ \mathcal{F}_r - \rho_m(V_r\dot{\gamma} + \gamma\dot{W}_r - \gamma W_r \underset{\sim}{\nabla}\cdot\underset{\sim}{V}) + \rho_m \gamma W_\theta^2 \lessgtr 0 \qquad (7a)$$

$$|\theta| \lessgtr |\bar{\theta}|,\ \mathcal{F}_\theta - \rho_m(V_\theta\dot{\gamma} + \gamma\dot{W}_\theta - \gamma W_\theta \underset{\sim}{\nabla}\cdot\underset{\sim}{V}) - \rho_m \gamma W_\theta V_r r^{-1} \lessgtr 0 \qquad (7b)$$

$$z \gtrless \bar{z},\ \mathcal{F}_z - \rho_m(V_z\dot{\gamma} + \gamma\dot{W}_z - \gamma W_z \underset{\sim}{\nabla}\cdot\underset{\sim}{V}) \lessgtr 0 \qquad (7c)$$

where the boosting force enters through $\dot{\gamma}$ and $F_\theta$ following (1) and (5a). The $\bar{\theta}$ is negative in the righthand cylindrical coordinates due to $V_\theta < 0$, (3b). The superscript dot abbreviates the material derivative $\partial_t + V \cdot \nabla$ which gives the continuity equation $\dot{n} = -\underset{\sim}{n}\underset{\sim}{\nabla}\cdot\underset{\sim}{V}$ of the mass density $\rho_m = nm_o$. The radial force $\mathcal{F}_r$ is simply the inward Lorentz force $F_r = \rho_q(V_\theta B_z - V_z B_\theta)c^{-1} \approx \rho_q V_\theta B_z c^{-1} < 0$ because of (3). Its magnitude decreases radially toward $r^0$ following the idea of weak focusing at the first stage. However, at the second stage $r > r^0$ the $|F_r|$ must increase through $B_z^E$ to match the increased $\gamma$ (Fig.(2)). Thus before calculating $\underset{\sim}{\bar{x}}$ we must consider the stability by perturbing $\underset{\sim}{\bar{x}}$ with the amount $\underset{\sim}{\tilde{x}}$ due to a writhing velocity $\underset{\sim}{\tilde{V}}$ away from the orbit, according to $\underset{\sim}{\tilde{x}} = \underset{\sim}{\bar{x}} + \int \underset{\sim}{\tilde{V}}dt$. Therefore, $\tilde{z} \gtrless \bar{z}$ iff $\tilde{V}_z \gtrless 0$ giving the following two consequences. (i) Since a writhing ion can be evenly energized $\dot{\gamma}(-z) = \dot{\gamma}(z) > 0$ due to (1) and (4a), then writing (7c) for the perturbation shows $\dot{\tilde{V}}_z$ always decelerated.

$$\tilde{z} \gtrless \bar{z} = 0,\ \tilde{V}_z \gtrless 0;\ \rho_m \gamma \dot{\tilde{V}}_z = \mathcal{F}_z - \rho_m \tilde{V}_z(\dot{\gamma} - \gamma\underset{\sim}{\nabla}\cdot\underset{\sim}{V}) \lessgtr 0, \quad (8)$$

where $\mathcal{F}_z = F_z$ is substituted by (4b). (ii) Since a cyclotron unlike a betatron does not accelerate particles by means of the Lorentz force, then

$$c\tilde{F}_\theta = \rho_q(\tilde{V}_z B_r - \tilde{V}_r B_z) \approx 0 \qquad \text{means } \tilde{V}_r < 0 \text{ since}$$

$\tilde{V}_r B_z \approx \tilde{V}_z B_r < 0$ using (6) for the odd function $B_r$ and (3a) for the positive function $B_z$. Working against (i) and (ii) simultaneously, a writhing ion is radially retarded and axially decelerated. Thus the larger the $\tilde{V}_r$, the larger the $\tilde{r}$. Consequently, the bigger the $|E_\theta^P|$, the bigger the $\dot{\gamma}$ and therefore the bigger the deceleration $\tilde{V}_z$. On the other hand cf. Fig(2), the radially advanced ion having not gained enough energy will be pulled back by a stronger $B_z^E$ until energized by a stronger $|E_\theta^P|$. To prove $\langle\tilde{V}_z\rangle \approx 0 \approx \langle\tilde{V}_r\rangle$ see (14a) for acceleration, (10) revolution.

Fig. (2) illustrates the following picture. Outside the several radial cavities, i.e., inside the so-called cyclotron dees or more appropriately Fig.(3a) the present case inside the pies, an ion is screened from electrical but magnetic forces by the metal walls of the pies. Thus no boosting $\dot{\gamma} = 0$ implies $\gamma \approx (1-(r\dot{\theta}/c)^2)^{-\frac{1}{2}}$ = constant iff $r\dot{\theta} \equiv \pm \bar{r}\Omega$. Therefore $r \equiv \bar{r} + \tilde{r}$ and $\dot{\theta} \equiv -\Omega + \dot{\tilde{\theta}}$ yield $\dot{\tilde{\theta}} \approx \mp \Omega\tilde{r}/\bar{r}$. By choosing the lower sign for the positive ion, the net radial force becomes

$$F_r + F_c \equiv qr\dot{\theta}B_z^E c^{-1} + m_o\gamma r\dot{\theta}^2$$
$$\approx -q\bar{r}\Omega B_z^E(\bar{r})[1+(\tilde{r}/\bar{r})(r\partial_r B_z^E/B_z^E)_{\bar{r}}]c^{-1} + m_o\gamma\bar{r}\Omega^2[1-(\tilde{r}/\bar{r})]$$
$$\equiv -q\bar{r}\Omega B_z^E c^{-1} I \cdot (\tilde{r}/\bar{r}) \lessgtr 0 \text{ for } \tilde{r} \gtrless 0 \qquad (10a)$$

$$\text{iff } I \equiv (r\partial_r B_z^E/B_z^E)_{\bar{r}} + 1 > 0;\ n + 1 > 0;\ n > -1 \qquad (10b)$$

where the first term in (10a) cancels by definition $\Omega(\bar{r}) \equiv (qB_z^E/m_o\gamma c)_{\bar{r}}$ and the second term in (10b) is written with the field index $B_z^E(r) \sim r^n$. Stipulating (24) we conclude that there exists a limit for decreasing $B_z^E(r)$ radially but no limit for increasing. Since $\dot{\theta}(\bar{r}) \equiv -\Omega$ is demanded for all $\bar{r}$ $B_z^E(r)$ must, like $\gamma(r)$, increase as r increases. According to (10b), at the first stage $\gamma \approx 1$ we can choose $B_z^E(r > r^0) \sim r^{-0.7}$ following the practical cyclotron experience; at the second stage $1 < \gamma < K$ we can freely choose $B_z^E(r > r^0) \sim r^n$ any positive $n > -1$, in order to match $\gamma(r)$ as closely as possible. $1 < \gamma < K$ is due to $B_z^E(r^0) \lessapprox K B_z^E(r^{Out})$. The larger the ion's rest-mass, the smaller the change of $\Omega(r)$. Because $\mathcal{E} \equiv m_o c^2\gamma \equiv T + m_o c^2$ gives $\gamma = (1 + T/m_o c^2) \leq K+1$ the larger kinetic energy $T \leq K\ m_o c^2 = K\ 3.7$ GeV for Helium. Clearly within the K-limit and a finite superconducting coil, the larger the $m_o c^2$ the larger the T; in addition to the extra advantage, the smaller the radiation loss $(T/m_o c^2)^4$. These are two advantages of the present heavy ion cyclotron. The third is given in the next section.

## Phase Stability and Universal Orbit

Due to the radially matched forces to be specified precisely below for a booster and a coil, the new cyclotron can accelerate the inertia $m_o\gamma$ and $|V_\theta| \equiv |r\dot{\theta}| << c$ at a suitably chosen constant $\dot{\theta} = -\Omega$ toward a radially increased kinetic energy $T(r) = ([1-r^2(\Omega/c)^2]^{-\frac{1}{2}}-1)m_o c^2 << T(\Omega/c)$. Such a new cyclotron combines the simplicity of a cyclotron with the advantage of a synchrotron. A ready implement is the principle of phase stability proposed by McMillian and Veksler independently utilized in synchrotrons. The acceleration phase $\theta^P(t) \equiv (\omega - p\dot{\theta})t$ is the phase that an ion is accelerated through p number of radial cavities per $2\pi$ radians. It turns out to be determined by the magnetron frequency $\omega$ as follows.

$$E_\theta^P = E_o(\Delta z)\sin(kr)\sin\theta^P(t);\ \pi/2 > (\theta^P(t),kr) > 0, \quad (11)$$

If the magnetron is operated at a higher frequency $\omega$ than the integer p times ion-revolving frequency $\dot\theta$, the acceleration phase has the stability provided by the positive slope $\partial_t \sin\theta^P(t) > 0$, $\pi/2 > \theta^P(t) > -\pi/2$. Since the mean-free-time $\Delta t$ of an ion revolving with $\dot\theta$ from one resonant cavity to the other is $|\dot\theta|\Delta t \approx 2\pi/P$ for p cavities per $2\pi$, then a slight undermatch of $\Delta t = 2\pi/p|\dot\theta|$ with the magnetron period $2\pi/|\omega|$, i.e. $\omega > p\dot\theta$, ensures that the ion be continuously accelerated in the positive boosting fields inside all the cavities. Since we require $\dot\theta(\bar r) = -\Omega =$ constant, then $|\omega| > p\Omega =$ constant is easily satisfied. An ion with $|\dot\theta| \gtrsim \Omega(r)$, arriving earlier at the resonant cavities at a certain radius r, is accelerated slightly faster, due to the positive field slope with respect to the time; but because of its becoming slightly heavier than $m_o\gamma(r)$ by gaining a bit more energy $\lesssim eV$, the ion having the angular velocity $|\dot\theta| \lesssim \Omega(r)$ finds itself late in arriving at the remaining cavities at r, and therefore receives less boosting energy. Such a natural balance makes the ion phase $\theta^P(t)$ migrate stably back and forth along the positive slope of the magnetron fields. Thus having chosen the constant frequency $\omega > p\dot\theta =$ constant for example p = 6, the phase stability is incorporated into (11) inside six resonators, that have been centrally fed from six anode cavities at $60^o$ apart inside the magnetron. One distinct advantage in adopting a magnetron having a TE mode, instead of many Klystrons having a TM mode, is that a single radiation source can form a standing wave at the constant $\omega$. This is technically known as the $\pi$ mode, when the major anode cavities are separated by the distance $d = \pi/k$ apart, inside the so-called rising sun magnetron, or the wire strapped or unstrapped magnetron. Whichever the magnetron may be, both the efficiency and the power level need to be improved beyond the present microwave capacity toward longer wave length $\lambda$ and better mode separation required here. Otherwise, the synchronized Klystrons having the $|\omega| > p\Omega$ can equally feed six resonators with (11).

In order to integrate nonlinear (1)(7) for ion orbit in a plane, one assumption $V_z = 0$ about the plane orbit is made to decouple the plane orbit (7a,b) from wobbling about the plane (7c). This assumption is valid since the ion possesses a decelerated and negligibly small writhing speed $|V_z c^{-1}| \ll |V_\theta c^{-1}|$, due to the strong mirror focusing discussed in the previous section. Thus, setting $V_z \equiv 0$ and $V_r \equiv \dot r$, $V_\theta \equiv r\dot\theta$, $\nabla \cdot V \equiv 0$ for a single point ion; replacing $\dot\gamma$ in (7b) with (1) and dividing (7a,b) with $(-\gamma\rho_m)$; we obtain as follows both the radially centrifugal and the tangentially Coriolis' accelerations.

$$r\dot\theta^2 = -r\dot\theta\Omega + \dot r\,\dot\gamma/\gamma + \ddot r \quad (12a)$$

$$2\dot r\dot\theta = -\dot r\Omega + \Omega V_{os}[1-(r\dot\theta/c)^2] - r\ddot\theta \quad (12b)$$

$$V_{OS}/c \equiv qE_\theta^P/\Omega m_o\gamma c \equiv E_\theta^P(r,t)/B_z^E(r). \quad (12c)$$

By definition (12c) the radial quiver velocity $V_{OS}$ is independent of $q/m_o$, the charge-mass ratio. Thus for a fixed $\Omega$, (12a, b, c) become independent of the species. This fact allows us to accelerate various species tunable to an identical $\Omega(r^O)$ along a universal orbit (Fig.(3b)). The time rate of an ion travelling on this universal orbit depends on the species and turns out to $qE_\theta^H/m_o c$ radian/sec. If $\dot r \geq 0$ and $\ddot r \leq 0$ respectively outside and inside shielded pies, then the orbit is precisely a staircase built on a plane spiral having a gradually reduced radius (Fig.(3c)). During either accelerations or revolutions, the fundamental requirement $\ddot\theta \equiv 0$ must be strictly satisfied. Since $\ddot\theta \equiv 0$ iff $\dot\theta \equiv$ constant, then putting $\ddot\theta$ to zero in the tangential Coriolis (12b) and dividing (12b) with $\dot r\Omega$ and then adding 2 to each side of (12b) we obtain that a bounded entire function L(r) is a constant

$$\text{constant} = 2(\dot\theta+\Omega)/\Omega = [V_{OS}(1-(r\dot\theta/c)^2)+\dot r]/\dot r \equiv L(r) \quad (13a)$$

according to the Liouville theorem and the special relativity $|r\dot\theta| \leq c$. Since on the left hand $\dot\theta$ and $\Omega$ are constants, then solving $\dot r$ in terms of the constant gives

$$\dot r = V_{OS}(1-(r\dot\theta/c)^2)/(\text{constant}-1) \quad (13b)$$

The constant is chosen to be zero by the other equation (12a). Thus

$$\dot\theta \equiv -\Omega \equiv -qB_z^E(r)/m_o c\gamma(r) \equiv \text{constant};\ B_z^E/\gamma \equiv \text{constant} \quad (13c)$$

$$\dot r \equiv -V_{OS}[1-(r\Omega/c)^2] \quad (13d)$$

Since during the shielded revolutions the limit $V_{OS} = 0$ yields from (12b) the limit $\dot r = 0$, then (13d) is also valid for the shielded revolutions, as one expects for an entire function of r. In other words, (13d) is valid for any instant. Now the constant $\dot\theta$ is to be simultaneously satisfied with the centrifugal (12a), which yields by substituting (13c) into (12a) that $\dot r\dot\gamma/\gamma + \ddot r \equiv 0$, or,

$$\ddot r/\dot r \equiv -\dot\gamma/\gamma;\ \ddot r \gtrless 0,\ \dot r \lessgtr 0,\ \dot\gamma > 0. \quad (14a,b)$$

The necessary and sufficient Coriolis acceleration $2\dot r\dot\theta$ follows from the following integrated result of (14a)

$$\dot r/c = \text{constant}\ \gamma^{-1} = \text{constant}\ [1-(r\Omega/c)^2]^{\frac{1}{2}} \quad (14c)$$

Being a simultaneous solution of (12a) and (12b), (14c) must equal (13d). Setting them equal, we therefore obtain by definition of $V_{OS}$(12c) the follwoing design criterion.

$$B_z^E(r) = B_z^E(r^O)[1-(r^O/r_M)^2]^{\frac{1}{2}}[1-(r/r_M)^2]^{-\frac{1}{2}} \quad (15)$$

$$E_\theta^P(r,t) = E_\theta^P(r^O t)\ [1-(r^O/r_M)^2][1-(r/r_M)^2]^{-1} \quad (16)$$

Here $r_M \equiv c/\Omega$ is the maximally attainable radius of an ion according to $r\Omega \leq c$ the special relativity. Note that the heavier the ion mass the smaller the $\Omega$, and therefore, the larger the $r_M$. Since $r \ll r_M$ for the estimate later, the Taylor expansion in $(r/r_M)$ can match the increased booster criterion (16) with the already increased booster field (11) by making the height $2\Delta z$ of the resonator $(2\Delta r \gg 2\Delta z > r\Delta\theta)$ flare out like Fig.(3d) a parabola horn.

$$\Delta z(r) = (\Delta z(r^O)/kr)[1 + a(r/r_M)^2] \quad (17a)$$

$$E_\theta^P(r,t) = E_\theta^P(r^O,t)\ [\Delta z(r)/\Delta z(r^O)]\sin(kr). \quad (17b)$$

Here the constant $a \gtrsim 1$ can be varied to give the best Padé fit between (11) and (15b) over the domain: $\pi/2 \approx kr > kr^O$, $k = \pi/2\Delta r$ and $kr_M \gg \pi/2$. Since

$$|E_\theta^P(r)/E_\theta^P(r^O)| \lesssim \gamma^2 = [1+(T/m_oc)]^2 \leq (1+K)^2 \quad (17c)$$

for $T \leq K\ m_oc^2$, then for various heavy ions $m_oc^2 \gtrsim 10$

GeV a small fa[illegible]or K seems to be adequate for

$$r \le r_M[K(2+K)]^{\frac{1}{2}}(1+K)^{-1}.$$ Since the contour of $B_z^E(r,r)$ produced by the coil is shaped like

$$B_z^E = B_{min}(r^0)\exp(K^2z^2)\{(r/r^0)^n H_< + [1-(r/r_M)^2/1-(r^0/r_M)^2]^{\ell} H_>\} \quad (18a)$$

then for $m_oc^2 \gtrsim 10$GeV a small factor K is adequate for

$$B_z^E(r)/B_z^E(r^0) \lesssim \gamma = 1 + (T/m_oc^2) \le 1 + K \quad (18b)$$

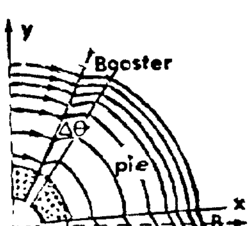
where $H_<$ and $H_>$ are symbols of the Heaviside function to separate the first stage $r < r^0$ from the second stage $r > r^0$. While the old cyclotron field index n is known to be $-0.6 < n < -0.7$, the new field index $\ell = -\frac{1}{2}$ is prescribed by the design criterion (15 ). Since the old field gives the known cyclotron orbit at the first stage, then only the new orbit at the second stage is derived below with the exactly integrable (13d) over $r_M \ge r \ge r^0$. In the limit $p\Delta\theta = 2\pi$ we obtain Fig. (3b) , where

$$r(t) = r_M[1-(r^0/r_M)^2]^{-\frac{1}{2}}\sin[A(t-t^0)] + r^0\cos[A(t-t^0)] \quad (19a)$$

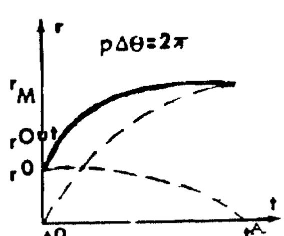
$$A = q|E_\theta^P(r^0)|/m_oc \quad (19b)$$

$$t_M^A - t^0 = [(\pi/2) - \sin^{-1}(r^0/r_M)]A^{-1} \quad (19c)$$

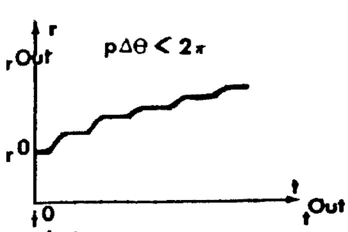
In the present case $p\Delta\theta < 2\pi$, the complete orbit consists of shielded revolutions ($\dot{r} = \ddot{r} = 0$) outside $p\Delta\theta$ and of boosted accelerations ($\dot{r} > 0$, $\ddot{r} < 0$) inside $p\Delta\theta$. This complete orbit Fig. (3c) is obtained by slicing vertically the continuous curve in Fig.(3b) into equal pieces of the length $\Delta t^A = \Delta\theta/\Omega$ due to the constant angular frequency $\Omega$ inside each booster of the angular width $\Delta\theta$, and then connecting each piece with a flat line of the length $\Delta t^R = (2\pi - p\Delta\theta)/p\Omega$ due to rotations $\dot{r} = \ddot{r} = 0$ with the constant $\Omega$ inside P shielded pies. A different species has a different time rate (19b). The total time span required for the complete orbit is bounded by the absolute maximum $t_M^A - t^0$ multiplied with the proportional factor $[(2\pi - p\Delta\theta) + p\Delta\theta]/p\Delta\theta = 2\pi/p\Delta\theta$.



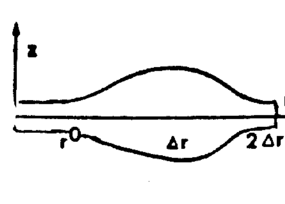

fig(3a)




fig(3b)




fig(3c)




fig(3d)

Finally note that the synergic focusing on the universal orbit by means of (i) $\partial_r|E_\theta^P| > 0$ and (ii) $\partial_r B_z^E > 0$ inside a booster has been proven in (14)(15) as well as implemented with (17)(18). Moreover, the synergic focusing on an ion is maintained when the ion revolves outside the booster, i.e. radially proven by(10) and to be axially proven by (21)(24) in the sixth section.

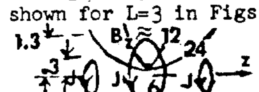
## Application and Computer Simulation

Intense beams of heavy ions accelerated to energies of 10 to 100 GeV might be used to ignite thermonuclear pellets and thus provide a basis for thermonuclear power generation.[4] For such a large scale application, the self fields of an intense beam can not be neglected during the acceleration. By integrating (2) an initial source $(c\rho, \underset{\sim}{J})$ yields the total field $(\underset{\sim}{E}, \underset{\sim}{B})$, then through (7) the field determines the accelerated fluid, which becomes a new source $(c\rho, \underset{\sim}{J})$ to be integrated again by (2) and so on. Such a bootstrapped tail-chasing requires an extensive and fast scheme of numerical iteration. Thus a vectorized code based on the general formulae (2,7) seems to be worthwhile implementing and then becomes readily accessible to simulate various cyclic accelerators including the present one, an isochronous cyclotron. Based on (2b) alone, the author has taken the initiative to write a simple vectorized code for the ASC computer at NRL to compute $\underset{\sim}{B}^E$. This $\underset{\sim}{B}^E$ is generated by a steady current flowing along the presently designed football coil defined by

$$x = R\cos\theta;\ y = R\sin\theta;\ z = Q[\theta/2\pi];\ 0 \le \theta \le 2\pi L \quad (20a)$$

$$R = r^C \mp |z-1|^G;\ 0 < z < 2 \quad (20b)$$

The gentle squeeze on a uniform coil is quantified herein (20b) with an index G, namely $G = \infty$ measures the infinite gentleness which means no squeeze on the constant radius $r^C$, since $|z-1|^\infty = 0$ for $0 < z < 2$. In (20b) the lower sign is adopted for a deflated football coil having a small central radius $r^C < r^M$'s $\le r^C + 1$ where $r^M$'s denote the mirror coil radii; the upper sign is chosen for a normal football coil having a large central radius $r^C > r^M$'s $\ge r^C - 1$. In (20a), $[\theta/2\pi]$ is numerically truncated by using the integer mode in programming, then (20a) describes precisely L number of circular loops having the interloop spacing $Q = 2/(L-1)$; if by using the real mode, (20a) describes a helix having the pitch Q. The results are schematically shown for L=3 in Figs (4a,b).




fig(4a)


fig(4b)

The other interesting results will not be presented here. To conclude, in viewing the generality of total electromagnetic field (2), orbit(7), and the increasing cost of accelerator hardware, it suffices to say that there is a future for accelerator simulation .

## Ion Self-Fields and Self-Focusing

We can numerically calculate the ion self-fields $(\underset{\sim}{E}^I, \underset{\sim}{B}^I)$ with the integral solution (2) consisting of both acceleration fields. Due to the negligible radiation loss in heavy ions, by averaging the Maxwell equations we can analytically deduce a certain useful formula for the following self-focusing:

$$\overline{B_r^I} \lessgtr 0,\ z \gtrless 0;\ \overline{F_z^I} \approx -\rho_q \overline{V_\theta B_r^I} \lessgtr 0,\ z \gtrless 0, \qquad (21a,b)$$

Three results follow for the averaged fields. (i) The Lorentz force along the ion beam direction is related, by the average specified with (22b), to the ion electric field $\overline{E_\theta^I}$ along the same direction.

$$\overline{F_\theta^I}\rho_q^{-1} \equiv (\overline{V_z B_r^I} - \overline{V_r B_z^I})c^{-1} = \overline{E_\theta^I} + \text{constant of } z \qquad (22a)$$

(ii) By averaging the Gauss law (2a) we have already shown that, $\partial_z \left|\overline{E_\theta^I}\right| \lessgtr 0$ for $z \gtrless 0$, (3d), $\left|\overline{E_\theta^I}\right|$ is maximal at the orbital plane. (iii) The average bar is consistently defined with a correlation time $\tau_z \equiv 2|\Delta z|c^{-1} > 0$ of $B_r^I(t, \underset{\sim}{x})$ such that

$$\overline{B_r^I} \equiv \tau_z^{-1} \int B_r^I dt = B_r^I(\underset{\sim}{x}) \qquad (22b)$$

is defined with the identically averaged Faraday law

$$\overline{B_r^I} = \tau_z \overline{(\partial_t \underset{\sim}{B}^I)_r} \equiv -\tau_z c \overline{(\nabla \times E^I)_r} \approx 2|\Delta z| \partial_z \overline{E_\theta^I} \lessgtr 0,\ z \gtrless 0 \quad (22c)$$

whereby the symmetry, $|\overline{E_z^I}|$ is neglected compared with $|\overline{E_\theta^I}|$, (3b). Consequently, based on (22c) the averaged Pynting flux points away from the plane, becomes by (3d) the minimum at $z = 0$, in agreement with the minimum radiation loss and with radiation recoiling of an ion back to the orbital plane,

$$\overline{(E^I \times B^I)_z} \approx -\overline{E_\theta^I}\,\overline{B_r^I} = -|\Delta z| \partial_z (\overline{E_\theta^I})^2 \gtrless 0,\ z \gtrless 0 \quad , \quad (22d)$$

Based on (22c) the magnetic flux $\overline{B}^I$ is shown by (3d) to be bulky and perpendicular to the plane and thus gives the self-focusing on the plane (21b). The deduction of (i) goes as follows. The Maxwell-Ampere law and the flux continuity when applied to the beam having $B_\theta^E \equiv 0 \approx B_0^I$ are respectively given as

$$(\nabla \times B^I)_\theta = -i\omega \epsilon(\omega) E_\theta^I;\ (\nabla \times B)_\theta = \nabla \cdot B^I = \nabla \cdot B^E = 0 \qquad (23a,b)$$

By means of partial integrations of (23a,b) it follows

$$B_r = \int [\partial_r B_z dz - \partial_z B_z dr] - i\omega \epsilon(\omega) c^{-1} \int E_\theta^I dz. \qquad (23c)$$

First we illustrate (23c) by proving for all r the external focusing on the plane. From the deflated football coil (17c) follow field inequalities

$$\partial_z B_z^E \lessgtr 0,\ z \gtrless 0;\ \partial_r B_z^E \gtrless 0,\ r \gtrless r^0;\ |\partial_z B_z^E| > |\partial_r B_z^E| \qquad (24a)$$

Then substituting these inequalities (24a) into (23c) yields for all $r \lesssim r^0$ the required inequalities for the external focusing on the plane

$$B_r^E \lessgtr 0,\ z \gtrless 0;\ F_z^E = -\rho_q V_\theta B_r^E c^{-1} \lessgtr 0,\ z \gtrless 0 \qquad (24b)$$

Next we exchange in (23c) the orders of $\partial_r, \partial_z$ with $\int dz$, $\int dr$; extrapolate $dz = V_z dt$, $dr = V_r dt$; integrate dt according to the time average. Having averaged by (iii), the longitudinal beam dispersion becomes valid according to the Maxwell-Ampere law alone (23a), namely the dielectric $\epsilon(\omega) = 0$ for $E_\theta^I \neq 0$. Consequently (23a) gives by the average $\partial_r \overline{B}_z^I = \partial_z \overline{B}_r^I$, then by replacing in (23c) the first term $\int \partial_r B_z^I dz$ with $\int \partial_z B_r^I V_z dt$

$$\overline{B}_r^I \approx \partial_z (\overline{V_z B_r^I} - \overline{V_r B_z^I})\tau_z \equiv 2|\Delta z| \partial_z \overline{F_\theta^I \rho_q^{-1}} \qquad (25a)$$

Equating $\overline{B}_r^I$ of (25a) to $\overline{B}_r^I$ of (22c) we have thus deduced (i). Such a tendency to stay on an orbital plane is natural for ions having a constant angular momentum. Due to the massiveness of ions, the negligible radiation loss $dL/d\theta$ does not change the orbital angular momentum $d_t(m_o \gamma r^2 \dot\theta) \approx 0$. This is not true for high energy electrons, but is for the heavy ions. The ion focusing on itself is based on the proper gradient of $|\overline{E_\theta^I}|$, (ii), giving the proper slope of $\overline{B_r^I}$, (iii). Since this gradient is not along the beam direction, the instability in $E_\theta^I$ itself is not affected by the present conclusion. However, to prevent any possible instability, the external boosters exert the angular forces $qE_\theta^P$'s on ions according to the principle of phase stability (11). Finally, in passing, note that the present result of self-focusing (21) is consistent with a recent finding[5] about a longitudinal beam propagation , because of no harmful neutralizing background plasma in an accelerator of the vacuum chamber pressure μ mm Hg.

The Nyquist theorem on the fluctuation and dissipation is well established for electric field fluctuations $\langle \tilde{E}_\theta^I(\underset{\sim}{x},\omega)^2 \rangle \propto (\mathrm{Re})Z(\underset{\sim}{x},\omega)$ in a current. Due to the ensemble average of ion-ion and ion-molecule collisions, $\langle \tilde{E}_\theta^{I\,2} \rangle$ becomes appreciable in the case of positive and real dissipation (Re)Z on the orbital plane. A recent study[5] shows binary collisions to be possible under $q\underset{\sim}{v} \times B_z^E c^{-1}$, iff $v_\perp = v_\perp^{(1)} - v_\perp^{(2)} = 0$. This conclusion has a positive impact to the present scheme: $\Omega$ = constant and $\gamma = \gamma(r)$, although $B_z^E$ is not globally uniform. The following picture is envisioned. A heavy ion $m_o\gamma(r)$ at $(r, |\theta|, 0)$ being constrained by $\gamma(r)$ = constant at r can wander off $\Delta t$ in the z-direction and come back hitting the other ion trailing behind at $(r, |\theta| - \Omega\Delta t + 2n\pi, 0)$. Since $v_\perp \equiv 0$ nullifies the effect of $B_z^E$, the famous Rutherford Coulomb-scattering predicts a peak contribution along the z-direction. Then, having zero angular momentum in zero impact parameter and small linear momentum in the side-by-side collisions wobbling along the $\hat\theta$ direction, we conclude that inelastic channels and Coulomb excitations $\sim (v_z/c)^2$ are negligible. Furthermore, since the largest possible $V_z$ is not large enough to escape the trapping of the mirror field, then the ion-ion collision is less detrimental to an intense beam of multi-Amp's current being accumulated in the storage ring at the end of last stage of acceleration.

## Conclusion

The presently designed isochronous cyclotron is implemented here with the superconducted football coil (20) and van resonators with flare height (11,17). It can accelerate various species of heavy ions. The heavier the rest mass $m_o$ of an ion the better the present scheme will be. Because a small number K is required in gaining the kinetic energy $T = Km_oc^2$ and sharing the following advantages based on He[illegible] $< m_oc^2 <$ 223 GeV $U^{238}$: (a) negligible radiation loss $<< (T/m_oc^2)^4$ = [illegible], (b) finite radial increase of the coil field $B_z^E(r) \lessapprox (K+1)B_z^E(r^0)$, (c) finite radial increase of the resonator field $E_\theta^P(r) \lessapprox (K+1)^2 E_\theta^P(r^0)$. Here the radial increases are compared with those conventional cyclotron fields $B_z^E(r^0)$ and $E_\theta^P(r^0)$ used for $T \lessapprox 1\%\ m_oc^2$ at the first stage of acceleration $r \leq r^0$. The presently proposed synergic focusing on heavy ions is proved radially by (10)(14), axially by (24)(21), and angularly by (11). Then various species tunable to an identical frequency $\Omega(r^0)$ are accelerated along an exact and universal orbit (19) having a gradually reduced radius $r^0 < r << c/\Omega$. This $r^0$ and this scheme are the beginning of an isochronous cyclotron for heavy ions, which has combined the simplicity of a cyclotron with the phase stability of a synchrotron.

Acknowledgment is given to Dr. P. Sprangle for suggesting a magnetron for the present scheme.

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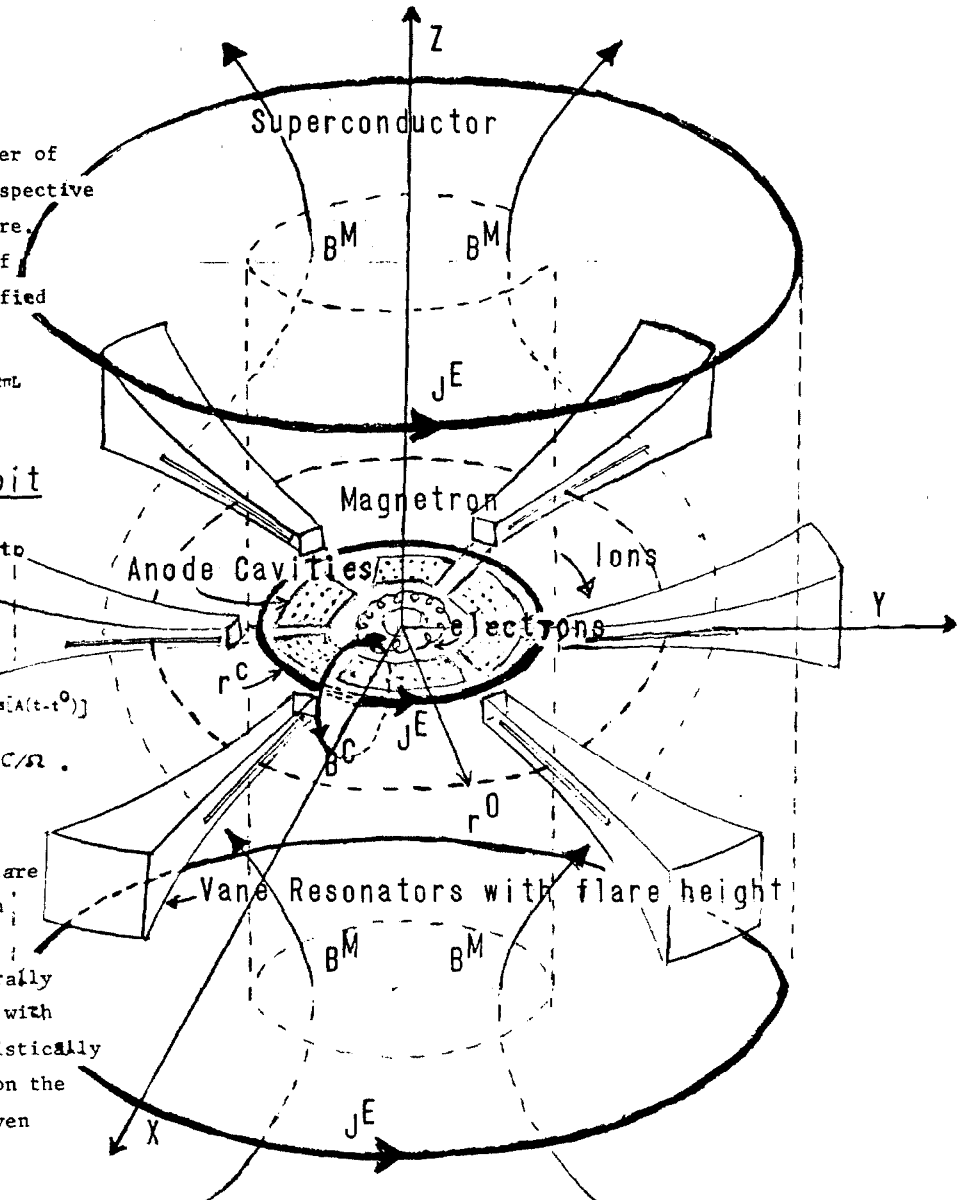
# An Isochronous Cyclotron*

$\Omega(r) = (qB_z/m_0\gamma c) = \text{constant}$




## (1) The Football Coil

The football coil has L number of turns, L = 3 is drawn for the perspective view in three dimensions shown here. The gentle squeeze on the radii of the tandem mirror coils is quantified with the index G. See Eq(20),

$$x = R\cos\theta;\ y = R\sin\theta;\ z = Q[\theta/2\pi];\ 0 \leq \theta \leq 2\pi L$$

$$R = r^C \mp |z|^G;\ -1 < z < 1\ .$$

## (2) Exact Universal Orbit

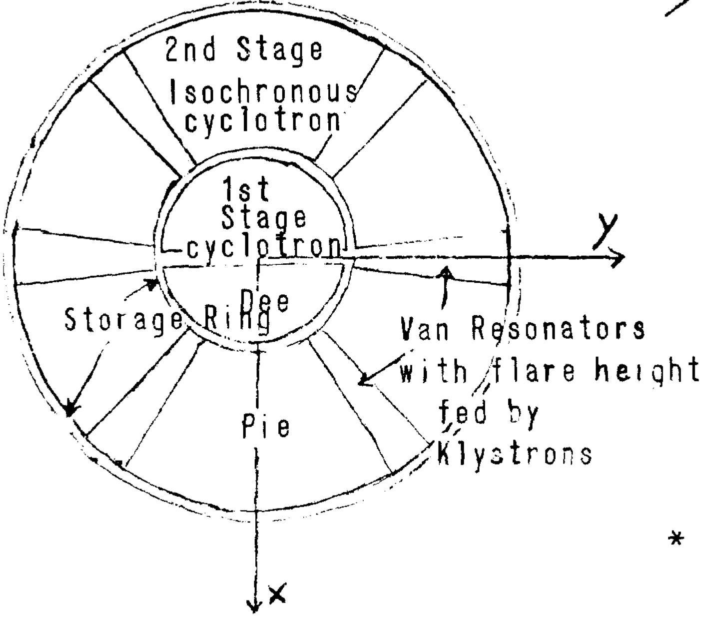
The various species tunable to an identical cyclotron frequency are accelerated along an exact and universal orbit(19),

$$r(t) = r_M[1-(r^0/r_M)^2]^{-\frac{1}{2}}\sin[A(t-t^0)]+r^0\cos[A(t-t^0)]$$

$$A = q|E_\theta^P(r^0)|/m_0c\quad ;\ r_M = C/\Omega\ .$$

## (3) Synergic Focusing

The van resonators with a flare height are fed with the radiation produced by either the centrally located Magnetron or the peripherally located Klystrons. This together with the football coil produce synergistically the focusing on ions travelling on the universal orbit. See the proof given in the text Eq(10,14,24,21,11).




## (4) Design Criterion & Parameter

For example 5 meters for 1 GeV $^4He^{++}$ ion in 10 K Gauss cyclotron field vacuum chamber pressure μ mm Hg.

$$B_z^E(r) = B_z^E(r^0)[1-(r^0/r_M)^2]^{\frac{1}{2}}[1-(r/r_M)^2]^{-\frac{1}{2}}$$

$$E_\theta^P(r,t) = E_\theta^P(r^0 t)\,[1-(r^0/r_M)^2][1-(r/r_M)^2]^{-1}$$

(a) It stores milli-Amp's into a ring at the end of each stage up to multi-Amp's intensity of ion current.

(b) finite radial increase of the coil field $B_z^E(r) \lessapprox (K+1)B_z^E(r^0)$, (c) finite radial increase of the resonator field [illegible] $(K+1)^2 E_\theta^P(r^0)$. Here the radial increases [illegible] compared with those conventional cyclotron fields [illegible] and [illegible] used for $T \lessapprox 1\%\ m_0c^2$ at the first stage of acceleration $r \cong r^C$.

* Navy Case No. 61,896