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LONGITUDINAL AND TRANSVERSE SPACE CHARGE LIMITATIONS ON TRANSPORT OF MAXIMUM POWER BEAMS*

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Summary

The maximum transportable beam power is a critical issue in selecting the most favorable approach to generating ignition pulses for inertial fusion with high energy accelerators. Maschke and Courant[1] have put forward expressions for the limits on transport power for quadrupole and solenoidal channels. We have included in a more general way the self consistent effect of space charge defocusing on the power limit. The results show that no limits on transmitted power exist in principal. In general, quadrupole transport magnets appear superior to solenoids except for transport of very low energy and highly charged particles. Longitudinal space charge effects are very significant for transport of intense beams.

Introduction

Since the prime consideration is constraining the beam within a given diameter, it is assumed that the focusing system consists of equally spaced elements of equal strength. In the case of quadrupole focusing, the elements are x or y focusing quadrupole magnets. For the solenoid, all elements are focusing. To simplify the discussion, we use thin lens approximation. In the quadrupole focusing, the distance between the focusing and defocusing lens is the half period L. The distance between the lenses in the solenoid focusing is the period 2L. It is furthermore assumed that the emittance in the xx' plane is equal to the yy' plane emittance (ϵ).

Quadrupole Focusing

In the quadrupole channel, the beam cross section is no longer circular but elliptical with semi-axes a and b. Neglecting the image forces, the equations of motion can be written in the form

$$\frac{d^2x}{ds^2} \pm \frac{qeGx}{Am\gamma\beta c} - \frac{N_1 q^2 e^2 x}{\pi\epsilon_o a(a+b)\, Am\gamma^3\beta^2c^2} = 0 \qquad (1a)$$

$$\frac{d^2y}{ds^2} \mp \frac{qeGy}{Am\gamma\beta c} - \frac{N_1 q^2 e^2 y}{\pi\epsilon_o b(b+a)\, Am\gamma^3\beta^2c^2} = 0 \qquad (1b)$$

where q = charge state of particle
G = quadrupole gradient
A = mass number
m = rest mass/nucleon
N_1 = number of particle/unit length

We see from these equations that the space charge defocusing in the x direction is a function of the beam dimension b in the y direction and vice versa. Starting the period in the center of an x focusing lens, the solutions of Eqs. (1) become

$$(x_n) = (M)(x_{n-1})$$

$$(y_n) = (m)(y_{n-1})$$

where

$$M_{11} = m_{11} = M_{22} = m_{22} = 1 - \frac{L}{f_1} + \frac{L}{f_2} - \frac{L^2}{2f_1f_2} \qquad (2a)$$

$$M_{12} = 2L + \frac{L^2}{f_2} \qquad (2b)$$

$$m_{12} = 2L - \frac{L^2}{f_1} \qquad (2c)$$

The quantity $\cos\mu = \frac{1}{2}(M_{11} + M_{22}) = \frac{1}{2}(m_{11} + m_{22})$ determines the focusing property of the system. From Eq. (2a), we find

$$\cos\mu = 1 - \frac{L}{f_1} + \frac{L}{f_2} - \frac{L^2}{2f_1f_2} \qquad (3)$$

The beam is diverging for $|\cos\mu| > 1$; for $\cos\mu \geq 1$, there is no net focusing. For $0 > \cos\mu > -1$, we have overfocusing, but the beam is still converging. If $\cos\mu < -1$, the overfocusing results in a diverging beam. In principle, one could allow $\cos\mu$ to be negative in the absence of space charge. The limit at $\cos\mu = -1$, however, is very abrupt with small changes of focusing. In view of possible magnet errors and the use of a thin lens approximation, we choose here to avoid overfocusing so that

$$0 \leq \cos\mu = 1 - \frac{L}{f_1} + \frac{L}{f_2} - \frac{L}{2f_1f_2} \leq 1$$

with $\cos\mu = 0$ in the absence of space charge defocusing. The focal lengths f_1 and f_2 can be written in the form

$$\frac{1}{f_1} = \frac{1}{f} - \frac{K}{L}\,\frac{2\lambda}{1+\lambda} \qquad (4a)$$

$$\frac{1}{f_2} = \frac{1}{f} + \frac{K}{L}\,\frac{2\lambda^2}{1+\lambda} \qquad (4b)$$

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$$\frac{1}{f} = \frac{qeG\ell}{Am\gamma\beta c} \tag{5}$$

$$K = \frac{N_1 q^2 e^2 L^2}{2\pi\epsilon_o a^2 Am\gamma^3\beta^2 c^2} \tag{6}$$

$$\lambda = \frac{a}{b}$$

and ℓ is the length of the quadrupole magnet.

The beam dimensions a and b are given by

$$a^2 = \frac{M_{12}}{\sin\mu}\epsilon \text{ and } b^2 = \frac{m_{12}}{\sin\mu}\epsilon$$

Thus,

$$\lambda = \frac{a}{b} = \left(\frac{2 + \frac{L}{f_2}}{2 - \frac{L}{f_1}}\right)^{1/2} \tag{7}$$

In the absence of space charge forces ($\cos\mu = 0$), Eq. (3) reduces to

$$L = \sqrt{2} f \tag{8}$$

Substituting Eqs. (4a, 4b, and 8) in Eq. (3), it is found that K can be written

$$K = \frac{(1+\lambda)[(2-\sqrt{2})\lambda + 2 + \sqrt{2}]}{4\lambda^2}$$

$$\left[\left(1 + \frac{8\lambda\cos\mu}{[(2-\sqrt{2})\lambda + 2 + \sqrt{2}]^2}\right)^{1/2} - 1\right] \tag{9}$$

A second expression for K in terms of λ is obtained from substitution of Eqs. (4) and (8) into (7)

$$K = \frac{(1+\lambda)[2+\sqrt{2} - \lambda^2(2-\sqrt{2})]}{2\lambda^2(\lambda - 1)} \tag{10}$$

Simultaneous solution of these two equations gives λ and K for any given value of $\cos\mu$. From these values, one can obtain f_1, f_2, a, b, etc. Introducing $\alpha = \ell/L$ as the fraction of the space occupied by quadrupole magnets, B = Ga as the pole tip field, $N_1 = K\pi\epsilon_o\gamma B^2\ell^2/Am$ and power $P = \beta c N_1(\gamma - 1)Amc^2$, we obtain

$$P = F(\alpha,\mu) \times 10^{15}\left(\frac{A}{z}\right)^{4/3}(\gamma - 1)(\gamma\beta)^{7/3}B^{2/3}\epsilon^{2/3} \tag{11}$$

where

$$F(\alpha,\mu) = K\left[\frac{2\alpha}{\sin\mu}\left(2+\sqrt{2}+\frac{2K\lambda^2}{1+\lambda}\right)\right]^{2/3}\pi\epsilon_o c^3\left(\frac{mc}{e}\right)^{4/3} \times 10^{-15}$$

Table I gives the value of λ, K, and $F(\alpha,\mu)$ as a function of μ for $\alpha = 0.5$ and $\alpha = 1$. Also included are the ratios R_a and R_b, which are the ratios of maximum and minimum beam size, a and b, with and without space charge defocussing. (See Fig. 1.) The growth of the effective emittance due to space charge is approximately

$$\epsilon_{eff} = \sqrt{R_a R_b}\,\epsilon \tag{12}$$

Table I

μ	10^o	30^o	45^o	60^o	80^o	90^o
λ	2.097	2.123	2.162	2.220	2.336	2.414
K	0.269	0.239	0.197	0.141	0.050	0
$F_{\alpha=0.5}$	7.62	3.30	2.12	1.29	0.40	0
$F_{\alpha=1.0}$	12.10	5.24	3.37	2.05	0.63	0
R_a	2.65	1.55	1.29	1.14	1.03	1
R_b	3.06	1.76	1.44	1.24	1.07	1
ϵ_{eff}/ϵ	2.85	1.65	1.36	1.19	1.05	1

Solenoidal Focusing

In a very similar manner, one can calculate the power that can be transmitted in a solenoidal transport system. In this case, for equal emittances in both planes (xx' and yy'), the beam cross section is circular. Neglecting the image forces, the equation of motion becomes

$$\frac{d^2r}{ds^2} + \left(\frac{qeB}{2Am\gamma\beta c}\right)^2 r - \frac{N_1 q^2 e^2 r}{2\pi\epsilon_o a^2 Am\gamma^3\beta^2 c^2} = 0 \tag{13}$$

where r = x (or y), B = axial field of the solenoid and a is the beam radius. We use the thin lens approximation in which all elements are focusing and the period is 2L. We again avoid overfocusing as in the quadrupole transport case so that $\cos\mu = 0$ in the absence of space charge. We introduce $\alpha = \ell/2L$ as the fraction of the space occupied by the solenoids where ℓ is the length of a solenoid and obtain the following result

$$P = F(\alpha,\mu) \times 10^{15}\left(\frac{A}{z}\right)(\gamma - 1)(\gamma\beta)^2 B\epsilon \tag{14}$$

where

$$F(\alpha,\mu) = \sqrt{2\alpha}(\cot\mu)\,\pi\epsilon_o\left(\frac{mc}{e}\right) \times 10^{-15}$$

R is the ratio of beam size with and without space charge defocusing and represents the growth in effective emittance due to space charge. (Fig. 1) It is given by $R = \sqrt{1/\sin\mu}$. Table II gives $F(\alpha,\mu)$ and R as a function of μ. As in the quadrupole case, two values of F are given ($\alpha = 0.5$ and $\alpha = 1$).

Table II

μ	10^o	30^o	45^o	60^o	80^o	90^o
$F_{\alpha=0.5}$	13.2	4.03	2.33	1.34	0.41	0
$F_{\alpha=1.0}$	18.7	5.7	3.3	1.90	0.58	0
R	2.4	1.41	1.19	1.07	1.008	1

Longitudinal Space Charge Effect

In linear approximation, the longitudinal space charge force on a particle in a radially restrained bunch can be written in the form

$$F_{\parallel} = \frac{3Nq^2e^2gz}{\pi\epsilon_o\gamma^2L^3}$$

where N = number of particles in the bunch
q = charge state of the particle
L = bunch length = $\beta c\Delta t$
g = geometrical factor, g = 1 + 2 ln b/a for a cylindrical beam of radius a in a circular pipe of radius b

To prevent the bunch length from increasing, a time dependent external electric field must be applied. For a linear time dependence, the peak value of this field must satisfy the relation

$$|E_p| > \frac{3Nqeg}{\alpha 2\pi\epsilon_o\gamma^2L^2} = \frac{8.64 \times 10^{-9}}{\alpha}\frac{Nqe}{\gamma^2L^2}g$$

where α is the fraction of the space that is occupied by the electric field.

For $N = 10^{14}$, q = 3, γ = 1.4, g = 2, L = 1 m, and α = 1/3, the peak value of the electric field would be $|E_p| > 2.6$ MV/m.

Conclusion

One can see from Tables I and II that no real limit on transmitted power or current exists. In practice, however, the increase in beam size and the effective emittance growth provide technical and economic limits. In addition, these results assume emittance matching to the channel acceptance at the beginning of the periodic transport structure for any transmitted current. The transition structure which can properly match the output of a synchrotron to the transport lines indicated above for a wide range of currents and space charge effects has not been designed.

Substitution of realistic beam characteristics into the power Eqs. (11) and (14) clearly indicates the superiority in transmitted power for the same effective emittance growth of a quadrupole transport system over a solenoidal transport system. The only exception for this conclusion is for transport of very low energy and high charge state beams.

Longitudinal space charge forces are significant in the transport of intense beams. Transport lines for such beams will require strong electric fields in the form of RF cavities or linear induction accelerators in order to maintain the longitudinal bunch length.

References

1. E. D. Courant, ERDA Summer Study of Heavy Ions for Inertial Fusion, Report No. LBL-5543, p. 72 (December 1976).

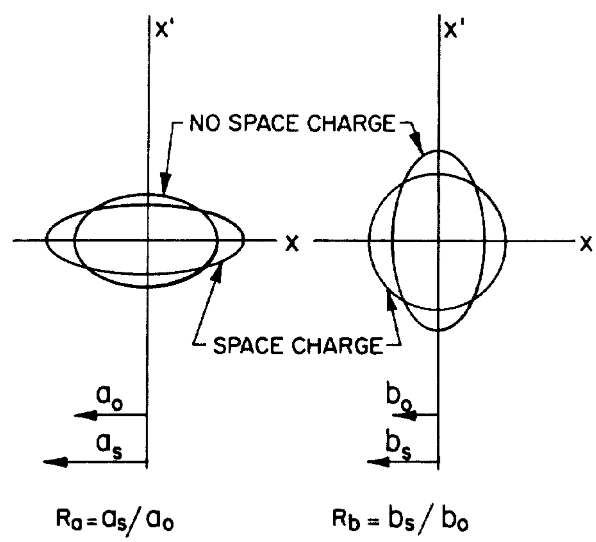

Fig. 1. Matched xx' phase space ellipses at the center of focusing and defocusing quadrupoles with and without space charge