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MORE PROPERTIES OF MIGMA ORBITS

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Summary

FEC is supporting a continuing investigation of the properties of high momentum, single particle orbits in an azimuthally symmetric 'mirror' magnetic field. Results have been obtained in several areas. Here we report calculations leading to a Hill's equation describing stable perturbations on midplane orbits in a field dominated as to shape by its quadratic terms. The results of some numerical tests are also presented, indicating that the equation may be of useful accuracy over an interesting range of parameters.

The field shape assumed is, in cylindrical coordinates

$$B_z = 1 - \alpha\lambda r^2 + 2\alpha z^2 \tag{1}$$

$$B_r = -2\alpha rz$$

and the equations of motion may be derived from the Lorentz force law

$$\vec{\ddot{r}} = \vec{v} \times \vec{B} \tag{2}$$

or from the Hamiltonian, which can be written

$$H = \frac{1}{2}(\dot{r}^2 + \dot{z}^2) + U_{eff}(r,z) = \frac{mv^2}{2} \tag{3}$$

$$U_{eff} = \frac{v_\phi^2}{2} = \frac{1}{2}\left(\frac{P_\phi}{r} - A\right)^2 \tag{4}$$

$$A = \frac{1}{r}\int_0^r r\, B_z\, dr \tag{5}$$

Here A is the magnetic vector potential (having only an azimuthal component) and p_ϕ is the (conserved) component of the angular momentum canonically conjugate to the azimuthal coordinate. Also, we have adopted units in which mass, m, and zero order ($\alpha = 0$) gyrofrequency, qB_0/mc, are set equal to unity. This means that r, z, $\dot{r}$, and $\dot{z}$ have units of length (e.g., $v_\perp = [\dot{r}^2 + v_\phi{}^2]^{1/2}$ becomes the zero order gyroradius of a midplane orbit.) The parameter α is typically given by $\alpha \sim 1/L^2$ where L is the radial dimension of the device and the quantity αv^2 in these units is dimensionless, being typically $\lesssim .1$.

One more preparatory item. The angular momentum turns out to be conveniently expressed in terms of an impact parameter, b, defined from

$$p_\phi = vb + b\, A(b,0). \tag{6}$$

This has at least two solutions which are denoted by b and b_c in order of their magnitude.

In order to deal with the orbits analytically, it proves helpful to make a change of independent variable from t to an angle, θ, which is a function of the coordinates and velocity components. A definition of θ has been found, for orbits of interest, with the properties that, first, θ advances smoothly with t (no wild variations in θ and, in particular, never $\dot{\theta} = 0$), second, θ corresponds to an intuitive notion of phase of radial oscillation (as distinguished from phase of gyration), and third, when only the linear terms in z and $\dot{z}$ are retained, θ reduces to a simple form, namely

$$\operatorname{Tan}\,\theta = \frac{-r\dot{r}\, B_z}{v^2 + r v_\phi B_z} \tag{7}$$

After a fair amount of algebra, one can solve approximately for $r(\theta)$ using (3) and (7) and thus remove r and $\dot{r}$ from the parameter space, leaving $\{(\theta,z,\dot{z})\}$ as the space in which we work. In actual fact, one only solves for $r(\theta)$ to specified order in α, treated as a smallness parameter. It should be noted that expansion of all relevant quantities in powers of α is assumed valid and convergent. Convergence, however, is far from guaranteed, let alone that the first or first few terms should suffice. Justification at this time rests simply on the fact that it appears to work.

The quantity $\dot{\theta}$ plays a central role, appearing as it does in

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} \tag{8}$$

Hence some effort has been expended to obtain an accurate form of $\dot{\theta}(\theta)$, starting from (7). The result obtained is

$$\dot{\theta} = -1 + \alpha\lambda[2v_c^2 + 2p_c + 2v_c\sqrt{v_c^2 + 2p_c}\,\operatorname{Cos}\theta + 2v_c^2\,\operatorname{Sin}^2\theta] \tag{9}$$

in which

$$v_c \equiv v(1 + \alpha\lambda v^2) \tag{10}$$

$$p_c \equiv bv_c + \frac{b^2}{2} + \alpha\lambda[4v_c^4 + 8bv_c^3 + 6b^2v_c^2 + 2b^3v_c]$$

If α is set to zero in (10), then (9) contains no terms of $o(\alpha^2)$ or higher. With α non zero in (10), it appears that the dominant part of the contribution of higher order terms is accounted for, primarily in p_c, which was obtained by substituting b_c for b in $p_\phi(b;\alpha{=}0)$. This has been tested numerically by integration of

$$\Delta t = \int_\theta^{\theta + 2\pi} \frac{d\theta}{\dot{\theta}} \tag{11}$$

and comparison to the period obtained by direct numerical integration of the equations of motion for midplane orbits at various energies and angular momenta. Without the 'correction terms' in (10), the deviations amounted to as much as 10% of the shape dependent part of $\dot{\theta}$ (the portion which vanishes when α does.) With correction terms, typical deviations amount to tenths of a per cent of the shape dependent part or parts in 10^4 of the total.

Of course not all cases were tested. For sufficiently large v and/or b it is known that (9) must fail. As the radial confinement limit is approached, we must have $\dot{\theta}\to 0$. For b = 0 and $\lambda = 1$ this limit corresponds to $v_m = .2722/\sqrt{\alpha}$ and $\dot{\theta}\to 0$ is not indicated by (9). Tests were performed at b = 0 with v in the range $.14/\sqrt{\alpha}$ to $.17/\sqrt{\alpha}$ and, at $v = .153/\sqrt{\alpha}$ with b in the range $\pm .1/\sqrt{\alpha}$. These ranges were chosen as being of particular interest for the migma experimental program.

Using (9) and (10) we may proceed to the linearized equation of motion for perturbations on the midplane orbit. From (3) and (4) we have

$$\ddot{z} = -2\alpha z\left(\frac{r^2}{2} - p_\phi - \lambda\alpha\frac{r^4}{4}\right) \tag{12}$$

and

$$z'' = (\ddot{z}/\dot{\theta}^2)-(z'/\dot{\theta})(d\dot{\theta}/d\theta) \tag{13}$$

Rather than obtaining z'' which will involve terms proportional to both z and z', it is convenient to use the standard transformation

$$z = w\ e^{\Phi} \tag{14}$$

with

$$\Phi = (-1/\dot{\theta})(d\dot{\theta}/d\theta) \tag{15}$$

so as to eliminate the first derivative term. It should be noted that z and w are quite 'similar' since Φ contains only terms of order α and higher.

After a good deal of algebra, we obtain, without the correction terms of (10),

$$\begin{aligned} 0 = w'' &+ w\{\alpha[2v^2+ (2+\lambda)(v^2+vb)\cos\theta - 2v^2\lambda\cos 2\theta] + \\ \alpha^2\lambda\ &[(20-\lambda)(v^4+v^3b+ v^2b^2/2) \\ &+ \cos\theta((41 + 2\lambda)\ \frac{v^4}{2} + (61 + 2\lambda)\frac{v^3b}{2} + 4v^2b^2+ 4vb^3) \\ &+ \cos 2\theta(\frac{\lambda}{2}v^4+ (8 + \lambda)\ v^3b+(4 + \frac{\lambda}{2})\ v^2b^2) \\ &+ \cos 3\theta(- \frac{5}{2} -\lambda)(v^4+v^3b) \\ &+ \cos 4\theta(\frac{\lambda v^4}{2})]\} + o(\alpha^2) \end{aligned} \tag{16}$$

When the correction terms of (10) are included, their effect is to add a term, T, to the r.h.s. of (16) where

$$\begin{aligned} T = \alpha^2\lambda w\{&\cos\theta[\frac{15v^4}{2} + \frac{19v^3b}{2} + 5b^2v^2+ vb^3] \\ &- 8\ v^4\ \cos 2\theta - \frac{9}{2}\cos 3\theta[v^4 + v^3b]\}. \end{aligned} \tag{17}$$

As will be evident momentarily, there is reason to believe that addition of this term results in accounting for the larger part of the α^2 dependence, but not all of it.

The parameter λ , when not equal to 1, requires a local source of magnetic field and might be thought of as a crude means of approximating the effect of a diamagnetic contribution. It is interesting, at any rate, to note that λ=0 (flat net field, radially) gives $\dot{\theta}\equiv-1$ and simplifies (16) drastically to

$$0 = w'' + w\alpha[2v^2+ 2\ (v^2+ vb)\cos\theta] \tag{18}$$

which is a simple Mathieu equation.

The general case of (16) is a Hill equation and not so simple. Techniques for dealing with a Hill equation are to be found in the literature, however, and McLachlan[1] gives a discussion in Chapter 6.

Noting that αv^2 occurs, in some cases, multiplied by rather large numbers (20 and 41), one is led to question whether the approach works at all. It might happen that many more powers of α are needed. To test this, we may calculate the curves in the (v,b) plane corresponding to the boundaries between zones of stable and unstable solutions and compare to results of orbit integration. The full calculation is somewhat painful unless computerized, hence only one point on the boundary of the first unstable zone is calculated.

When Hill's equation is written in the form

$$y'' + y\ [\theta_o + \sum_{n=1} \theta_{2n}\ \cos 2n\theta] = 0, \tag{19}$$

the solutions are known to have the form

$$y \pm = e^{\pm\mu} \sum_{n=-\infty}^{\infty} C_{2r} e^{\pm 2in\theta} \tag{20}$$

and the complete solution becomes linearly unstable. Using techniques discussed in McLachlan, we find the critical value of v at which this first occurs (for b=0, λ=1) to be $v_{crit} = .2215/\sqrt{\alpha}$, without correction terms, and $v_{crit} = .2182/\sqrt{\alpha}$ with. This is to be compared to a determination of v_{crit} by direct integration of orbits, yielding $v_{crit} = .2158/\sqrt{\alpha}$. The deviations amount to 2.6% and 1.1% respectively.

Clearly, this leaves some room for improvement, and efforts are under way in that direction. But this level of accuracy, if maintained for other calculations related to the solutions of (16), will serve quite well for most purposes. It may be worth noting that the value of v_{crit} just calculated turns out to be 80% of the maximum which can be confined (at b=0) in a field of the shape given by 1) with λ=1 (external field only.) Thus a reasonable expectation exists, to be tested, that 'most' stable perturbations on the midplane orbit are adequately described by the Hill equation prescribed by (16) and (17).

It should be noted that the near midplane orbits corresponding to $v>v_{crit}$ are not adequately described by (16). Most such orbits are linearly unstable and enter regions of large, z, z' so that the nonlinear terms become important. Numerical integrations done by several researchers indicate that such orbits are typically bounded. These integrations seem to indicate that, over a wide range of parameters, orbits are bounded by invariant tori in the $\{(z,\dot{z},\theta)\}$ space. That is, tori with the property that an orbit initiated on the surface of the torus remains forever on the surface and those initiated inside (outside) the torus remain forever inside (outside). It should be emphasized that this conclusion, while strongly indicated, is inductive rather than deductive. Preliminary calculation of the parameters of the tori relevant to orbits in the first unstable zone have been completed (to be reported elsewhere), and the value of v_{crit} obtained therefrom is .2159, in excellent agreement with the numerical results and that obtained from the Hill equation.

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References

1. N.W. McLachlan, "Theory and Applications of Mathieu Functions," Clarendon Press, Oxford, 1947.