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COMA - A LINEAR MOTION CODE FOR CYCLOTRONS

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Summary

COMA, a computer program written in FORTRAN IV, accurately calculates the relative motion of up to 5000 particles over the complete energy range of the TRIUMF cyclotron at a cost about 1% of our numerical integration codes. In addition it can simulate the action of diagnostic probes and slits. COMA uses linear transformation matrices, obtained from an equilibrium orbit code, that include coupling between radial and longitudinal motion. The energy gain and vertical focusing forces at the dee gap are given by an improved analytical model.

Introduction

It was decided to supplement our numerical integration codes by one that inexpensively calculated particle motion in the TRIUMF[1] cyclotron by multiplying an initial vector in six-dimensional phase space by the appropriate linear transformation matrices obtained from our equilibrium orbit code. Although the use of transfer matrices to describe ion-optical systems is well known and many synchrotron laboratories use this technique,[2] the use of matrix methods to describe motion in a cyclotron, while not unknown,[3,4] is less common. Possible reasons for this are that the usual chief regions of interest are injection and extraction, which are less amenable to matrix techniques, and in between particles make only 100-200 turns. Also, beam time is sufficiently inexpensive that diagnostic techniques can be developed on the actual cyclotron. In contrast, particles in TRIUMF make more than 1200 turns, including hundreds of turns in regions where the action of field imperfections are important. Also, since TRIUMF is capable of supplying beam to several experiments simultaneously, beam development time is correspondingly more valuable, and it is worth while developing a simulation program to describe the action of the machine tuning parameters and diagnostic equipment.

Theory

Let the displacement of a particle from its equilibrium orbit at some azimuth, θ, be given by a vector V_θ

$$V_\theta = \begin{pmatrix} x \\ p_x \\ \psi \\ z \\ p_z \end{pmatrix}_\theta = \begin{pmatrix} R-R_{eo} \\ P_r-P_{r_{eo}} \\ \phi-\phi_{eo} \\ Z-Z_{eo} \\ P_z-P_{z_{eo}} \end{pmatrix}_\theta \qquad (1)$$

where R,Z and P_r,P_z are the radial and axial co-ordinates and momentum, ϕ is the phase relative to the RF, and EO refers to the equilibrium orbit. COMA assumes that the motion about the EO is linear, and the vector V_2 at any azimuth θ_2 can thus be obtained by multiplying the initial vector V_1 by the transfer matrix M_{12} for particles of that energy between θ_1 and θ_2, provided that the angles θ_1 and θ_2 do not include a dee gap crossing. Dee gap crossings can be handled if one assumes an instantaneous energy gain and if the angle at the end of one transformation and the beginning of the next lie on the dee gap. When a dee gap is encountered, the energy is changed, and the co-ordinates of the particle R,P_r,ϕ,Z,P_z with respect to the centre of the cyclotron will yield a new displacement vector V with respect to the EO for the new energy. Matrix multiplication then proceeds as before until the next dee gap is encountered.

The matrices are obtained from our EO code at a set of discrete energies; the matrix for a desired transformation at any particular energy is obtained by interpolation.

Calculation of Transfer Matrices

Our equilibrium orbit code CYCLOP is based on the work of Gordon and Welton[5] with several modifications of interest here. Our magnetic field expansion includes asymmetric as well as symmetric components, and we search for equilibrium orbits in both axial and horizontal planes.[6] We also integrate an additional linearised equation of motion to give the coupling between longitudinal (phase) motion and horizontal motion.

From Ref. 5 (Eq. 3.5a) we have, in cyclotron units, for a particle moving in a horizontal plane

$$dt/d\theta = \omega_o/\omega_{ion} = \gamma R/Q \qquad (2)$$

where $\omega_o = \omega_{rf}/h$, h being the harmonic number, $Q = (P^2-P_r^2)$, R being particle radius and P the momentum, and γ is the usual relativistic quantity.

$$\text{Then} \quad d\phi/d\theta = h\left[(\omega_o-\omega_{ion})/\omega_{ion}\right] = h\left[(\gamma R/Q)-1\right]. \qquad (3)$$

Substituting the relationships from (1) into (3), and retaining the linear terms, yields

$$\frac{d\psi}{d\theta} = \frac{h\gamma x}{Q} + \frac{h\gamma R_{eo}\, p_x\, P_{r_{eo}}}{Q^3} \qquad (4)$$

In CYCLOP this Eq.(4) is added to the linearised equations for horizontal and axial motion about the EO which have had included in the field expansion additional terms $B_R(R,\theta)$, $B_\theta(R,\theta)$ and $\partial B_z/\partial z(R,\theta)$ defined in the 'plane of measurement' of the axial component $B_z(R,\theta)$.

For each of a set of energies CYCLOP determines the equilibrium orbit, then retraces the EO integrating in addition the linearised equations for two rays V^A and V^B. The elements of the transfer matrix M_{12} between θ_1 and θ_2 are obtained from the solution of

$$V_2^{A,B} = M_{12}\, V_1^{A,B}$$

$$\text{where} \quad V_1^A = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_1^B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (5)$$

CYCLOP is run at a series of energies, T, corresponding to a radial interval of about one-quarter of the magnet gap; the field properties are felt to vary slowly between these steps. CYCLOP outputs the matrix elements and equilibrium orbit properties $(T,R,P_r,Z,P_z,\phi)_{eo}$ at pre-determined azimuths or it can write them onto a tape at each Runge-Kutta (R.K.) step. This tape can be subsequently read by another program which interpolates in the data to give values at any angle, not necessarily an integral number of R.K. steps, that may be desired later. In the absence of field imperfections it is more accurate to calculate the EO properties and matrices over the basic symmetry angle, rather than over a full revolution, and obtain values at other azimuths by multiplication.

Energy Gain and Electric Focusing

In previous work[7] the energy gain and axial focusing at dee gap crossings have been calculated assuming a constant electric field component normal to the dee gap and equal to the field at the centre of the gap (y=0). For symmetric dee gap geometries, where the vertical gap is comparable to $g_c(r)$, a more realistic field $\mathcal{E}_y$, normal to the gap, is given by

$$\mathcal{E}_y = \frac{V_d}{g_c}\left(\cos\left(\frac{\pi y}{g_c}\right) + 1\right) \qquad (6)$$

where V_d is the dee voltage and $g_c(r)$ is taken to be the FWHW of the $\mathcal{E}_y$ obtained from a three-dimensional relaxation calculation of the electric field based on the TRIUMF dee geometry.

Including the time variation of the electric field but assuming a constant particle velocity, the energy gain on crossing the gaps is

$$\Delta T = \frac{qV_d}{g_c}\int_{-g_c}^{g_c}\left(\cos\left(\frac{\pi y}{g_c}\right) + 1\right)\left(\cos\left(\frac{hy}{r}\right) + \phi_c\right)dy \qquad (7)$$

where ϕ_c is the particle phase with respect to the RF at y=0. Integration yields

$$\Delta T = 2qV_d\left(\frac{\sin\left((hg_c)/r\right)}{(hg_c)/r}\cos\phi_c\right)\left(1 + \frac{h^2g_c^2}{(\pi^2r^2-h^2g_c^2)}\right). \qquad (8)$$

For our dee geometry in the range 12" < r < 40" the constant electric field model differs from the results of numerical integration through the three-dimensional electric field[8] using TRIWHEEL by 10-20%, while (8) differs by only 1-2%. If we assume different velocities V_1 and V_2 for each half of the dee gap crossing, a more complex expression gave results differing from (8) by $\lesssim$0.1% for our situation.

We now estimate the axial focusing using this same model for the electric field; the field components $\mathcal{E}_y$ and $\mathcal{E}_z$ are sketched in Fig. 1 for a static situation.

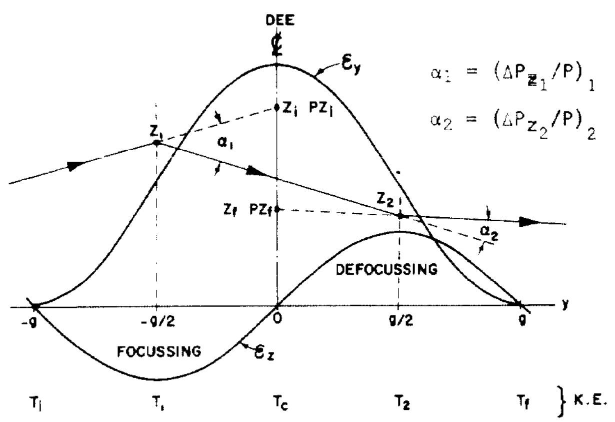

Fig. 1. The static electric field components, based on Eq.(6). The arrows indicate the motion of a typical ray crossing the dee gap; simple geometry gives the net change in Z and P_z.

The vertical impulse for a particle entering the dee gap at time t_i, crossing the plane y=0 at time t=0 and leaving the gap at time t_f is

$$\Delta P_z = q\left(\int_{t_i}^{0}\mathcal{E}_z\cos(wt+\phi_c)dt + \int_{0}^{t_f}\mathcal{E}_z\cos(wt+\phi_c)dt\right). \qquad (9)$$

$\mathcal{E}_z$ can be obtained to first order using div $\vec{\mathcal{E}}$=0 and assuming $\partial\mathcal{E}_x/\partial x$=0.

If for each half of the dee gap we assume an 'effective' axial displacement, radius of curvature, and particle velocity (Z_1,r_1,v_1), (Z_2r_2,v_2), then using $p_1v_1 = 2T_1$ (non-relativistic), we have for each half of the dee gap

$$\left(\frac{\Delta P_z}{P}\right)_{1,2} = \mp\frac{Z_{1,2}qV_d}{g_cT_{1,2}}\left(\frac{\cos\phi_c + \cos\left(\phi_c - (hg_c)/r_{1,2}\right)}{1 - \left((hg_c)/\pi r_{1,2}\right)^2}\right). \qquad (10)$$

The 'effective' $(Z,r,T)_{1,2}$ are approximated to be those of the static situation at $y=\mp g_c/2$ where, in addition, we localize the instantaneous impulse $(\Delta P_z/P)_{1,2}$. Thus, using (6), we have $T_1 = T_i + \left((\pi-2)/2\pi\right)V_d$ and $T_1 = T_2 + \Delta T - \left((\pi-2)/2\pi\right)V_d$ while $z_2 = z_1 + (\Delta P_z/P)_1 g_c$ and for the centre of TRIUMF $r_{1,2}(\text{in.}) = 18.79\ (T_{1,2}(\text{MeV}))^{\frac{1}{2}}$.

COMA Algorithms

The cyclotron conditions common to all particles are read in and stored. This includes up to 8 sets of transfer matrices and equilibrium orbit properties over the energy ranges of interest, dee voltages, as a function of radius if necessary, and the parameters determining the optics in the first gap crossings, together with the location of slits and probes, etc.

The particles starting conditions may either be read in, perhaps from the output of a previous run, or chosen to populate a six-dimensional phase space as follows. The horizontal and vertical emittances are specified in terms of ellipse semi-axes and tilt with respect to the x- or z-planes. The centres of the ellipses may be positioned anywhere with respect to the appropriate equilibrium orbit (EO), or alternatively the bundle centre can be specified in terms of absolute R, P_r and T rather than with respect to the EO. The input phase range is divided into equally-spaced intervals and presently it is assumed that the emittances are the same for each phase. Since COMA includes the optics of the injection gap[1] we can start particles before the gap where this is usually the case. The ellipses are populated uniformly, the 'phase space area' represented by each particle being an input parameter. The 'central ray' particle is always included and is run first. The option exists for random population of the emittances and a specified energy spread.

The particle parameters updated are energy T, phase ϕ, x, p_x, z, p_z, R, P_r, Z, P_z. Acceleration or deceleration proceeds by taking each particle in turn, testing to see if it is at a dee gap, or slit, or other special locations, then multiplying by the appropriate transformation matrix. At the dee gap the absolute coordinates R, P_r, Z, P_z, φ are calculated, the axial impulse and energy change determined by the methods described above, and we determine the EO parameters and matrices for the new energy. The vector V is calculated with respect to the new EO and motion continues as before.

The slits, probes and flags specified for TRIUMF are described in ref. 1. We consider a flag to be an obstruction in R,Z space that removed particles from the beam. The program can take account of up to 10 flags, identified by their azimuth and their spatial co-ordinates. A slit is described by two adjacent flags at the same azimuth. Each time a particle reaches a flag the parameters are stored for possible subsequent output and the next particle is chosen for acceleration.

A simulated probe head can be comprised of up to 3 radial and 20 vertical sections making up a grid in R-Z space. It can be moved outward for up to 1000 steps from some initial radius. A particle is accelerated until its radius exceeds the probe initial radius whereupon a count is registered in a bin corresponding to the grid element hit; in addition one other parameter, e.g. Z or ϕ, can be binned. The probe is then moved to its new position and the particle tested to see if it still hits the probe; if so it is binned in the element associated with the new position. If the particle does not hit the probe in the new location it is accelerated until it does, and the cycle is repeated. The total number of particles in a bin or group of bins corresponds to the current.

A moving finger corresponds to a narrow probe or wire producing a shadow on a current-reading probe at larger radius; it has one radial section and up to 20

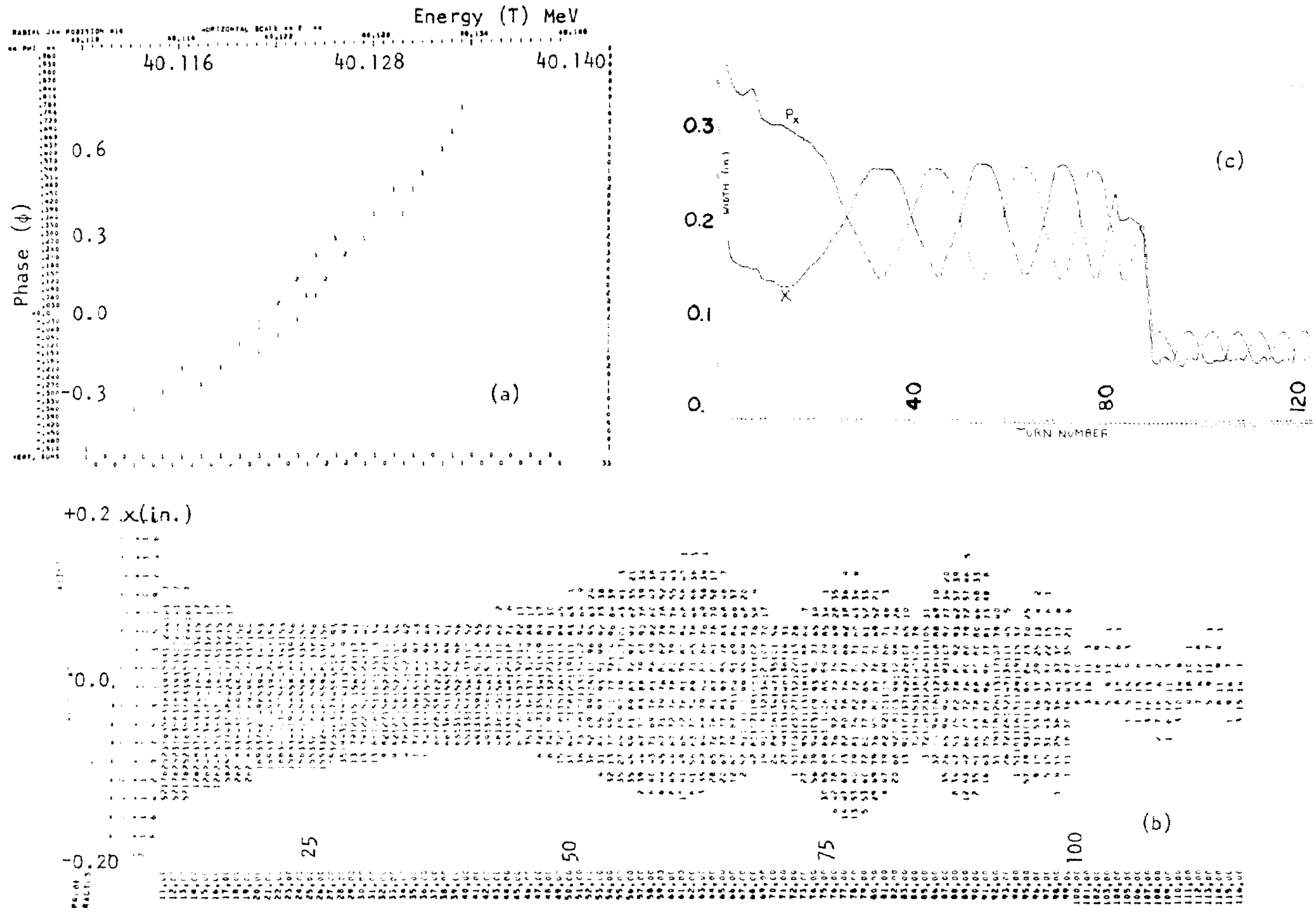

Fig. 2. Simulated effect of a beam (similar to that in Fig. 3) passing through 0.2 in. wide slit at 28 in. radius and a set of slits 0.07 in. wide at 96 and 99 in. radius (30 MeV).
(a) Transmitted emittance at 40 MeV projected in T-ϕ space; note phase width of 1.25 deg.
(b) Current distribution seen by a probe as a function of X and R.
(c) Width of X and P_x as a function of turn number, maximum width after all slits is 0.08 in.

vertical section. Each time an accelerated particle crosses the finger azimuth we determine, from the particle radius, which if any of the up to 1000 finger positions is being hit. If this is the first impact for this particle at this position, we bin the desired parameter.

Output

For detailed ray-tracing the particle parameters may be printed out at any or none of the azimuths. We can display the population distribution in any one parameter against any other parameter, e.g. Fig. 2a; these can be produced for the beam hitting a flag or at any desired turn number. For the first two radial divisions of a probe head the current distribution of any parameter binned in the vertical bin can be printed as a function of radius, see Fig. 2b. We can also plot the envelope or the 'width' of any two parameters as a function of turn number, see Fig. 2c. The sum of all vertical grid elements in the radial divisions of a probe head is plotted and printed as a function of probe radius and in terms of a percentage of total starting and surviving current, Fig. 3, the latter for the case where the beam may have passed through a slit.

Tests

We have checked the assumptions made in COMA against our general orbit codes GOBLIN and TRIWHEEL. GOBLIN traces particles through a magnetic field and is correct to second order axially but has a primitive model for dee gap transit time effects. TRIWHEEL integrates particles through a three-dimensional electric field obtained from a relaxation code.

Comparisons of COMA and GOBLIN static runs at 5 MeV for particles of different displacements, up to 2 in., from the EO showed that for small displacements the differences are comparable in magnitude with those seen between GOBLIN runs with different Runge-Kutta step sizes and may be ascribed to round off. For larger x-displacements with z=0 the maximum difference in R is 0.012 x^2; if x=0 and z=1 in. it appears as if ν_z is changed by 0.0025 ν_x; and x-z coupling causes similar changes for x≠0.

We accelerated a particle with 0.25 in. radial amplitude for 165 turns from 40 to 104 MeV, and the difference in final position given by the two codes was 0.001 in., the difference in phase 0.3 deg.

We would also expect linearity problems when ν_r=6/4[9] at 435 MeV and 298 in. radius. Several particles with up to 2 in. radial amplitude have been accelerated for 150 turns, passing through this resonance for several different magnetic fields. Generally the detailed motion agreed; however, for one field, known to be highly non-linear, large differences in (x,p_x), of up to 50% of the peak amplitude, were seen at the resonance. In all cases the final radial emittances given by the two codes were very similar but rotated slightly, the overlap being better than 90%. While the effective ν_r can differ by 0.008 close to the resonance for particles of different amplitudes, the vertical motion is well

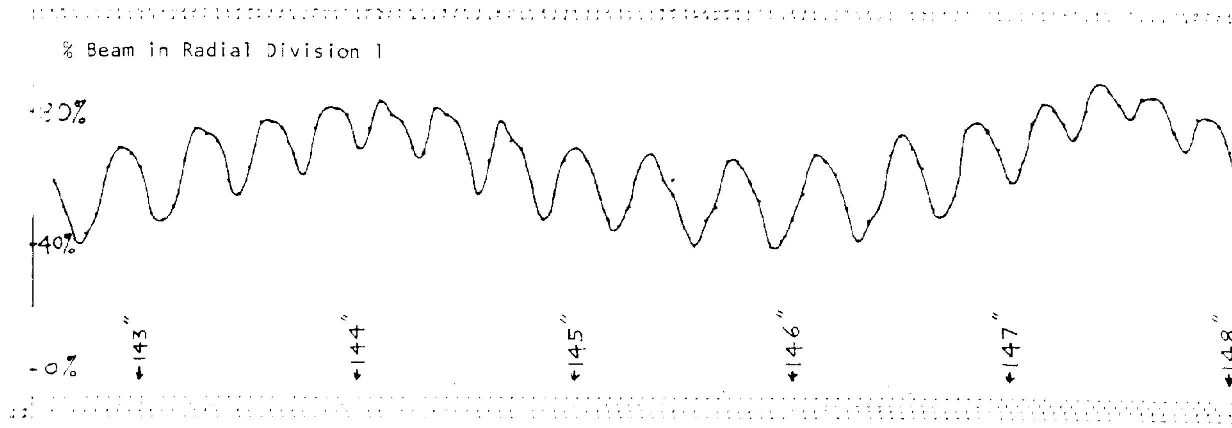

Fig. 3. Simulated turn pattern seen on a 0.2 in. wide differential probe and produced by a beam of 18 deg phase width with an initial radial emittance of 0.25 π in.mrad injected 0.1 in. off centre.

behaved, differences in ν_z being 0.0001.

In the central region the electric field nonlinearities are more important than those of the magnetic field. For a simple dee geometry they can be ignored if the beam axial excursions do not exceed 50% of the axial aperture. A radial ellipse at a typical phase 17.5 deg was accelerated from just after the injection gap and the results compared with TRIWHEEL at the 8th turn; the ellipse overlapped by 90%. The present TRIUMF dee gap design produces some field components parallel to the gap as well as perpendicular; these can be taken account of by appropriate additional changes in radial momentum at a dee gap crossing.[10]

Figs. 4a,b compare the analytic energy gain and vertical focusing impulse, expressed as $\Delta\nu_z^2 = \Delta p_z/\pi z$, with the results of numerical integration for a severe case corresponding to the first gap crossing after the injection gap. The agreement is excellent. Several particles of different phase have been run backward from 3 MeV, and the vertical amplitude given by the two codes differed by ≤10% at 0.5 MeV; this agreement can be extended through the injection gap by properly parameterizing the radial perturbations caused by electric field distortions at the centre.

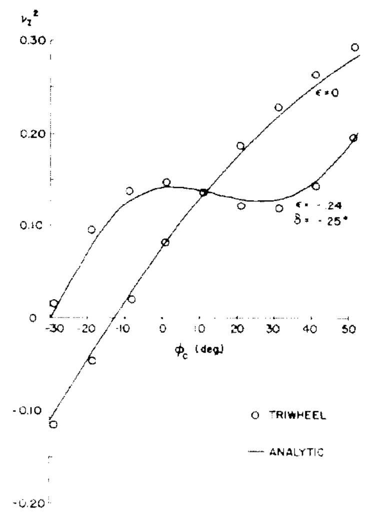

Fig. 4a.

Comparison of the electric contribution to ν_z^2 of the first dee gap crossing given by (10) and by numerical integration. We consider the case of a third harmonic superimposed on the fundamental wave form

$V_d = V_0\,(\cos\omega t + \varepsilon(\cos 3\omega t + \delta))$ and of the fundamental alone (ε=0).

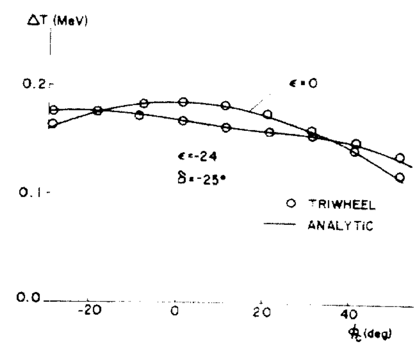

Fig. 4b.

Comparison of the energy gained for the same case as Fig. 4a.

Conclusions

We conclude that the linear and other approximations used in COMA are sufficient to give a useful description of the relative motion of a large number of particles at all energies in TRIUMF, and that, over most of the cyclotron for reasonable betatron amplitudes, the description of the absolute motion is as accurate as our general orbit codes and at 1% of the cost. The code has been used for ray tracing in conjunction with other codes and for study of emittance-dependent effects, misalignment of cyclotron components and action of diagnostic equipment. An interactive version has been developed using a Tektronix 4023. Future plans include analytic calculation of changes in matrix elements caused by trim coil changes, incorporation with other codes describing the injected and extracted beam lines, and possible development of a computer-controlled tune-up procedure.

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