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EFFECTS IN LOW PERIODICITY LATTICES RESULTING FROM LOW β INSERTIONS

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Introduction

In intersecting proton storage rings such as the existing ISR and the proposed ISABELLE,[1] the beams are stacked in momentum space and occupy a momentum bite of 1-3%. The chromaticity of the rings reflects a dependence of the betatron tune as well as the structure function,[2] β, on momentum. Correction of these effects can at least in part be accomplished by a distribution of sextupole magnets. The momentum spread within which the variations of the tune and the β-function with momentum can be eliminated or tolerated is the momentum aperture.

When the rings consist of a series of regular cells, the horizontal dispersion function is non-zero everywhere and it is possible to correct for the effects of chromaticity by sextupoles placed at or near each focusing element. This is the case in the ISR where the momentum aperture in fact exceeds the available physical aperture. In a high performance storage ring such as ISABELLE, it is required for the interaction regions to be contained in long straight sections, i.e. experimental insertions. These are matched to the regular lattice with respect to β and dispersion functions. A great deal of effort has in the past been devoted to designing such matched insertions to fulfill experimental requirements.[1] Recently they have been reexamined from the point of view of available momentum aperture and they have been found to be not entirely satisfactory. The presence of large numbers of non-locally-correctible focusing elements in these rather complex straight sections give rise to a prohibitively small momentum aperture.

In this paper, a much simpler type of experimental insertion will be considered. Both the bending necessary to accomplish the beam crossings and the dispersion function matching will be done in the regular lattice by a modification of the bending in the cells adjacent to the straight sections. The insertion itself will contain only one focusing unit on each side of the crossing point. Results of numerical calculations on the off-momentum effects for this type of insertion will be presented. It will be shown that a reasonably large momentum aperture can be obtained over a certain range of values of the parameters characterizing the insertion and ultimately determining the storage ring performance. The consequences of the added chromaticity correcting sextupoles with respect to both limitations on the betatron aperture and restrictions on tune will also be given.

2. General Considerations

The introduction into a cell-lattice structure of long straight insertions means (1) a low periodicity to the structure function; specifically, the periodicity is the number of identical insertions, and (2) a stronger sensitivity of the structure function to momentum, which we consider explicitly as a function of p as well as the physical azimuth, θ, measured with respect to a reference point on the equilibrium orbit. Since the tune for a period is related to β by $\nu=(R/2\pi)\int_0^{2\pi} d\theta/\beta$, where R is the average radius for the period, then it is also momentum dependent.

To the extent that the momentum dependence of ν is linear, this chromatic effect can be compensated by a 0th azimuthal harmonic sextupole distribution. Since an off momentum particle has a displaced equilibrium orbit because of the presence of radial dispersion, a sextupole distribution inserted at locations where such a dispersion exists, simulates a quadrupole with effective gradient,

$$G = S\, x_p(\Delta p/p) \quad , \tag{2.1}$$

where S is the sextupole strength, x_p is the local dispersion and $\Delta p/p$ is the momentum deviation from the central or "matched" momentum. There are three reasons why local chromatic correction suggested by (2.1) is not an appropriate procedure. First, it is desirable from the point of view of storage ring performance to have $x_p = 0$ in the straight sections near the beam collision region. Second, since the chromatic effect is large in the insertions, large local sextupole components are required. These not only excite third integer resonances, but can induce higher order effects, such as 4th order resonances. These latter are excited, albeit more weakly, even by a more distributed sextupole distribution than would result from local correction. And third, effects due to higher order in $\Delta p/p$ are not corrected by this procedure. It appears that trying to maintain a reasonable dispersion function in the long straight sections with quadrupoles only, coupled with the introduction of large sextupole components, tends to strengthen these higher order terms.

We will therefore consider a lattice structure in which the dispersion function is brought to zero in the insertions by appropriate removal of bending in two contiguous normal cells. The chromaticity is corrected with sextupoles placed in some of the cells. The insertion β-functions are matched to the cells at some central momentum. We will consider in Section 3 a set of insertions which can essentially be characterized by two parameters, ℓ, the distance from the crossing point to the first focusing element, and β^*, the value of β at the crossing point. We will show that to first order in $\Delta p/p$, and for low values of β^*, the momentum distortion of the β-function has a roughly linear dependence on the ratio, ℓ/β^*. For sufficiently small values of this ratio, the sextupoles correct the working line slope properly, producing only a small quadratic dependence of ν on momentum. However, as ℓ/β^* increases, effects nonlinear in momentum become more evident, appearing as curvatures in plots relating both β and ν to momentum. These are studied numerically using the SYNCH computer program, with insertion matching at the central momentum done with the TRANSPORT program.

The β-function distortion with momentum, although generally considered as a 1/2 integer structure resonance effect, is, strictly speaking, not a resonance effect at all. It is true that as the period tune approaches a 1/2 integer value, the period becomes unstable, with β rising towards infinity. However, we are concerned with β-function distortions occurring when the tune is 1/4-integer, the maximum distance from the two closest resonant values. In fact, even at period tunes of 1/4-integer, lattices with low β insertions tend to have significant β-variations with p. The choice of period tune is thus constrained to be in the vicinity of 1/4-integer.

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The β-function variation with p is not the only factor limiting the choice of period tune. Because of the long field-free regions, we might anticipate that required nonlinear field distributions or systematic nonlinear field errors will excite nonlinear structure resonances, further limiting the period tune choice. An example already alluded to is the necessary sextupole distribution for chromaticity correction. This gives rise to 1/3 integer structure resonances, and because of its substantial magnitude, also excites a 1/4-integer resonance (which is 2nd order in the sextupole strength). These sextupole structure resonances will be discussed in section 4. Although systematic field errors in the magnets will contribute to nonlinear structure resonances, they can in principle be corrected by special correcting coils within each magnet. On the other hand, certain nonlinear fields are intrinsic to the design of the storage rings. The sextupole field necessary to correct for chromaticity is one of them. In addition, an octupole field distribution may be required to compensate for working line curvature,[3,4] arising for example, from the beam space-charge or from the off-momentum effect mentioned above. This will give rise to a 1/4-integer structure resonance, one which is 1st order in the strength of the octupole excitation field. Another nonlinear force present is the beam-beam force. This highly nonlinear force excites all even ordered structure resonances, in particular the 4th, 6th, 8th and 10th. The limitations on period tune are thus severe: we must be near 1/4 integer but not too near; we must be far from 1/3 integer, but not near 1/6 or 1/5. We are thus left with a small region between 1/4 and 1/5, say 0.78. It is interesting to observe that the classic argument for high periodicity is especially applicable in this situation: We would like to have a certain freedom to choose a machine tune to avoid constructional-error-induced-resonances and to allow for a sufficient tune spread to provide Landau damping for transverse coherent instabilities.[4] However, we would also like to be able to accomplish this essentially independent of the period tune. Of course, this implies a high periodicity. For our purposes here, we will assume that the periodicity is identical to the number of experimental insertions. The specific number is irrelevant since we are concerned only with structure effects. We ignore at this time the possibility of a more complex arrangement with more than one insertion per period.

In general, the physical aperture determines the maximum allowable momentum spread in a stack. The ISR, for example, has a physical aperture corresponding to a momentum aperture on the order of ± 2%. For machines operating at a higher energy than the ISR, it might be thought that for a given stack current, such machines would operate with momentum apertures appreciably smaller than the maximum determined by the physical aperture, this following from the adiabatic damping of $\Delta p/p$. (It is Δp that is invariant for coasting beams.) A small beam momentum spread is of course a very desirable condition because of the distortion of the structure function for off-momentum orbits. However, the smaller momentum aperture conclusion is not an obvious one. There is in fact a required minimum momentum spread. The criterion which determines this required minimum spread is related to the maximum allowable impedance for longitudinal stability of a coasting beam.[5] (We assume that there is no limitation in our choice of density-i.e. current per unit momentum bite-which implies a measure of control over the density at the source as well as control over potential dilution during the complex process of getting the beam to the storage ring and then stacked.) For machines with equal current and similar momentum distributions, the impedance maximum, $|Z|_{max} \propto p/E_{tr}^2$, where E_{tr} is the transition energy. Since the transition energy is to a large extent determined by the circumference, we have roughly, $|Z|_{max} \propto p/C^2$. This means that a higher energy translates into a smaller required momentum spread and a larger circumference translates into a larger required momentum spread. (In the latter case, it is probably more difficult to control the impedance sources around the larger circumference, and so the momentum spread required would go up even further.) For any given machine, the minimum momentum spread required can be found from the full expression for the limiting impedance,[5]

$$\left|\frac{Z}{n}\right|_{max} = \frac{p\eta}{I}\left(\frac{\Delta p}{p}\right)\left(\frac{\Delta p}{p}\right)_t \quad , \qquad (2.2)$$

where I is the average current, n is the azimuthal harmonic number, $\Delta p/p$ is the full width at half-height of the momentum distribution, $(\Delta p/p)_t$ is the full width of the distribution tail, $\eta = 1/\gamma_{tr}^2 - 1/\gamma^2$, γ is the energy and γ_{tr} is the transition energy, the last two quantities having units of proton mass.

3. Momentum Dependence of the Structure Function and Tune-General Formulation of 1st Order β Distortion

An expansion of the β function in powers of momentum yields[2] for the fractional β change, to 1st order in the momentum error, $\Delta p/p$,

$$\frac{\Delta\beta(\psi)}{\beta} = -\frac{1}{2}\frac{|J_2|}{\sin\mu_L}\left(\frac{\Delta p}{p}\right)\cos\left(2\psi + \delta - \mu_L\right) \quad , \qquad (3.1)$$

where $\mu_L = 2\pi\,\nu_L$ is the phase advance for the total period ν_L is the period tune, and for a discrete gradient distribution, J_2, a complex function, is given by

$$J_2 = |J_2|\,e^{i\delta} = \sum_{\ell} \beta_\ell\, q_\ell\, e^{i2\psi_\ell} \quad . \qquad (3.2)$$

Here, β_ℓ is the value of β at the ℓth quadrupole, q_ℓ is the integrated gradient ($q_\ell = \int G ds/p$), $\psi_\ell = R\int d\theta/\beta$ is the phase to the ℓth quadrupole element from the first quadrupole, implying that $\psi_1 = 0$, and ψ is the phase from a reference point to the first quadrupole. Note that all the azimuthal dependence of the J_2 function is implicit in the quadrupole counting. Thus, to find J_2 as a function of azimuth, we must keep reordering quadrupoles as we move across them, thus changing the relative phases and thus the value of J_2.

Low β Insertion

Consider a series of low β insertions matched to the lattice cells as shown in Fig. 1. Note that only the vertical β is focused to a low β value to avoid excessive β-values in the focusing triplet. This procedure is quite acceptable, since with horizontal beam crossing at angles of interest, we can presume the attainable luminosity to be independent of horizontal β. Because the contribution from cell quadrupoles from cell to cell tend to cancel, the total cell contribution to the β-distortion will be small compared to the contribution from the insertion quadrupoles. Exact cancellation results for the case of an even number of cells with $\pi/2$ phase advance. Both these conclusions remain valid for the chromaticity correction sextupoles when they are properly located in the cells. We thus calculate the effect of a low-β insertion alone. Of course higher order effects will be complicated by the interference of the cell contribution with the insertion contribution.

If the β function goes to a low value, it will rise rapidly. As it reaches the focusing system, we can assume it to be sufficiently large that there is little phase advance across the triplet. The triplet can therefore be considered as a simple focusing lens. Its required strength can be estimated from the needed change in slope of the β function to match it to the cells. It can be deduced that the change in slope leads to an effective triplet strength, given approximately by

$$(\beta q)_{\text{effective,triplet}} = (\ell|\beta^*)\left[1 + (\beta^*/\beta_{\text{cell}})^{\frac{1}{2}}\right], \quad (3.3)$$

where β_{cell} is the minimum β for the cell. Therefore, since the phase advance between the two focusing systems across the low β region must be close to π, then their contributions to the 1st order β-distortion add. We thus obtain $J_2 = 2\,(\beta q)_{effective}$, which leads to the β-distortion outside the double-triplet-region, $\Delta\beta/\beta = (\beta q)_{effective}\,(\Delta p/p)\cos(2\psi-\mu_L)/\sin\mu_L$, where ψ is the phase from the observation point to the first triplet, and μ_L is the period phase advance. For a 1/4-integer period tune, the distortion effect to 1st order is zero between the two triplets, since $J_2 = 0$ in this region. Outside this region, and in particular, in the normal cells, the distortion modulates at a frequency of twice the betatron tune, with a maximum given by

$$|\Delta\beta/\beta|_{\max} = (\ell/\beta^*)\left[1 + (\beta^*/\beta_{\text{cell}})^{\frac{1}{2}}\right](\Delta p/p)/\sin\mu_L \quad (3.4)$$

We show in Fig. 2 the function $|\Delta\beta/\beta|_{max}$ versus $\Delta p/p$ for the various insertions seen in Fig. 1, defined by various values for ℓ and β^*. Along with these theoretical straight lines, we show the results of a computer analysis. The fit is good for positive $\Delta p/p$, but the effect becomes stronger than 1st order rapidly at negative $\Delta p/p$. This asymmetry results from the fact that a first order effect for negative $\Delta p/p$ is to increase the effective focusing of the triplet, which because of the phase relationships between various elements, does not occur for positive $\Delta p/p$. Since the dominant 1st order parameter is just the effective focusing strength of the triplet, it is not surprising that higher order effects occur at a lower magnitude of $\Delta p/p$ when $\Delta p/p$ is negative. In Fig. 3 we show the chromaticity corrected ν vs p curves for the 4 cases. Shown in the accompanying table are the values for the central tune of the period and the uncorrected chromaticity in each case. The influence of higher orders in $\Delta p/p$ is clearly evident at $\Delta p/p = \pm 0.5\%$. The results suggest the potential need for octupoles or higher order terms to compensate for this curvature. This is especially important in order to avoid a "brick wall" effect due to the transverse resistive wall instability.[3]

4. Nonlinear Structure Resonances

The sextupoles needed to compensate the chromaticity of the linear lattice can induce structure resonances if the phase advance in a period is a 1/3 integer multiple of 2π. However, in addition to these well known 1/3-integer sextupole resonances, if 2nd order terms in the sextupole field strength are taken into account, it turns out that 1/4-integer structure resonances can also be excited by a sextupole field distribution. We can explain this latter effect qualitatively by considering the one-dimensional equation of motion, $x'' + K(\theta)x + S(\theta)x^2 = 0$, where $K(\theta)$ is the gradient forcing function for the lattice and $S(\theta)$ is the azimuthal sextupole distribution function. Because of the nonlinearity, the betatron motion, in terms of the displacement variable x, will contain the frequencies 2ν, 3ν, ... as well as ν. Since $S(\theta)$ contains all integral Fourier components, the term Sx^2 contains, among others, terms of frequency $[k-\nu-(2\nu)] = k-3\nu$, having strength proportional to the (2ν) component in $x(\theta)$. Now when this frequency equals ν, i.e. $k = 4\nu$, then resonance results. Note that the effect is at least 2nd order in the sextupole strength since the (2ν) component of $x(\theta)$ is proportional to at least one power of the sextupole strength.

The strength of the sextupoles is fixed by the chromaticity of the linear lattice. If the chromaticity is $\xi = p\partial\nu/\partial p$, then the requirements for the sextupole strength function is given by, $\xi_x = (1/2\pi)\int S\,\beta_x x_p\,ds$; $\xi_y = -(1/2\pi)\int S\,\beta_y x_p\,ds$, where x_p is the horizontal dispersion function and the integral is to be taken along the equilibrium orbit around the circumference. Thus, for chromaticity corrected lattices, the uncorrected chromaticity is in fact a measure of the strength of the sextupole distribution. The resonances excited by this sextupole distribution, depending on various Fourier components of $S(\theta)$, have strengths which can be expressed in terms of the amplitude at which the resonance produces instability for a given deviation of the small amplitude tune from the resonant value. We therefore have that the resonant strength, i.e. the resonant amplitude, is related to the uncorrected chromaticity. Exact expressions for this relationship can be derived. In one dimension, we can write for the 3rd order stability limit,

$$a_3^2/\beta \approx (x_p^2/\beta)_{av}\,[(\nu-m/3)/\xi]^2/F_3 \quad (4.1)$$

and for the 4th order,

$$a_4^2/\beta \approx (x_p^2/\beta)_{av}\,[(\nu-m/3)(\nu-p/4)/\xi^2]/F_4 \quad . \quad (4.2)$$

Here, $2\pi\nu$ is the phase advance per period, ξ is the chromaticity per period, a_3 and a_4 are the resonant amplitudes at an azimuth where the structure function has the value β, and m and p are the integers closest to 3ν and 4ν, respectively. The quantity F_3 is a numerical factor of order unity, multiplied by the ratio of the (3m)th harmonic of $S(\theta)$ to the fundamental component of $S(\theta)$; while, F_4 has the form, $F_4 \approx \Sigma\, C_k S_k S_{p-k}/S_0^2$, where S_k is the kth harmonic of S, and the C_k's are certain numerical coefficients depending on the detailed nature of the sextupole distribution. We find that F_3 and F_4 are generally less than 1. For two-dimensional motion, we may expect similar expressions, and for the purposes of order-of-magnitude estimates, we will simply use (4.1) and (4.2), assuming F_3, $F_4 \lesssim 1$. Since, for reasons discussed in Sections 2 and 3 one tends to choose ν close to 1/4 integer, the 4th order stability limit is the important one. Thus, restricting ourselves to this case, and taking the ISABELLE parameters $x_p \approx 2$m, $\beta \approx 40$ m, ν(period) ≈ 2.75 (m=8, p=11), $\xi \approx 5$, and assuming $F_4 \approx 1$, we obtain $a_4 = 0.12\,(\nu-11/4)^{\frac{1}{2}}$ meters. For $\nu-11/4 \approx .0007$, $a_4 \approx 3.2$ mm.

Numerical computations for a lattice with insertion IV, as previously described, demonstrate the nature of the 1/4 integer sextupole resonance. The unstable fixed point agrees in order-of-magnitude with the estimate we have made. The phase space topology (vertical projection) near the 4th order fixed point is shown in Fig. 4 for tunes close to the 1/4 integer. An example of an unstable trajectory coupled in the horizontal and vertical planes is given in Fig. 5, with the vertical and horizontal projections shown in Figs. (5a) and (5b), respectively.

5. Conclusions

We have considered some of the limitations to the introduction of matched insertions into a cell-lattice structure by studying a set of simple insertions characterized essentially by the effective strength of a focusing triplet. Our results indicate that the maximum allowable strength may be much less than has previously been assumed. However, in at least one of our examples, a momentum aperture on the order of 2% as required for ISABELLE seems attainable. We also considered the impact of structure resonances on the choice of central tune. As well as exciting a 1/3 integer structure resonance, the necessary sextupoles for chromaticity correction also excite a 1/4 integer resonance. Using insertion IV, we have shown some of the characteristics of this resonance; in particular, an unstable phase space topology even for very small vertical amplitudes.

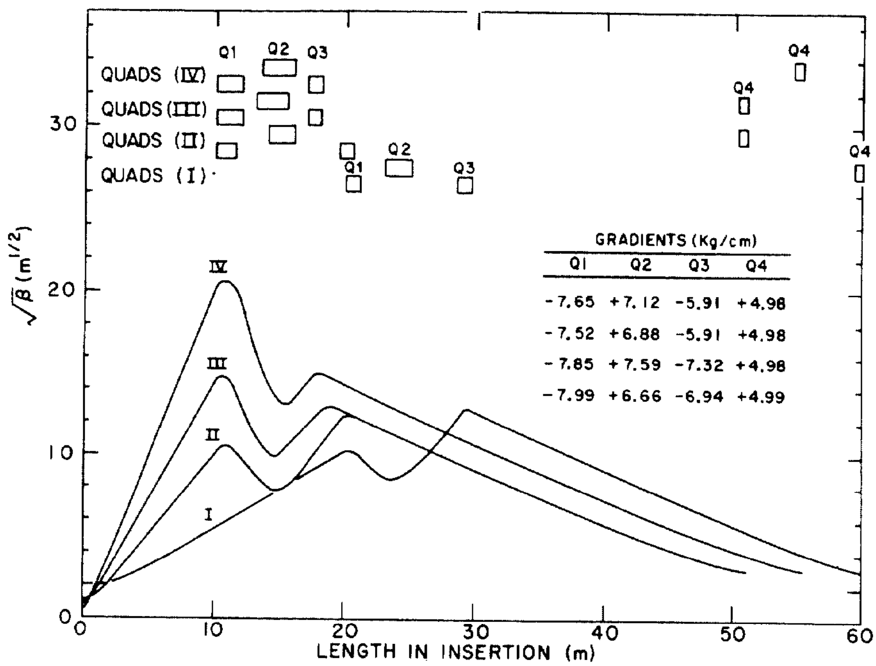

Fig. 1. β-function for low-β matched insertions. $\beta_{Cell} = 9.0$ m. Q4 is a lattice cell quadrupole.

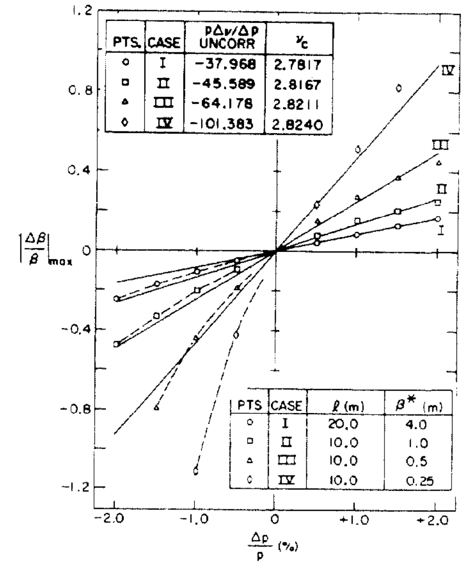

Fig. 2. $|\Delta\beta/\beta|_{max}$ vs $\Delta p/p$. Chromaticity corrected in normal cells. The period phase advance, $\nu_c \sim 3/4$ integer. Straight lines are theoretical estimates. Numerical results from SYNCH.

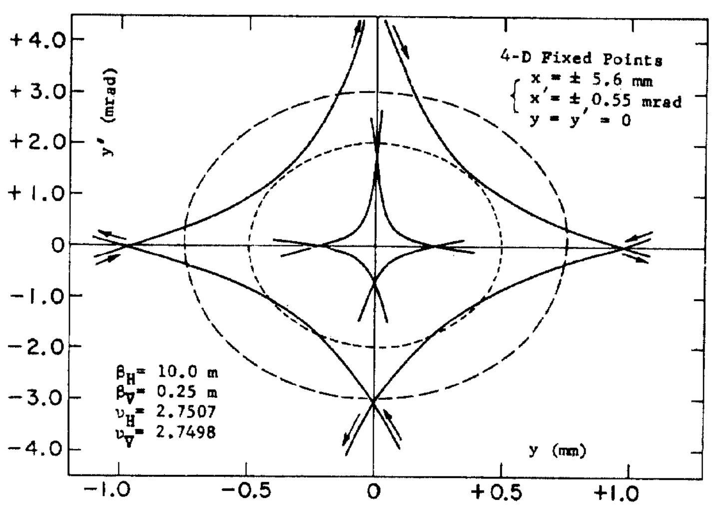

Fig. 4. Phase space topology near 4th order fixed point. Vertical projection. 4th order sextupole resonance. Elliptical plots are non-resonant trajectories. Corresponding motion in [x,x′] projection remains near fixed point. The apparent "trajectory crossings" are not fixed points, but are artifacts of the 2-dimensional projection.

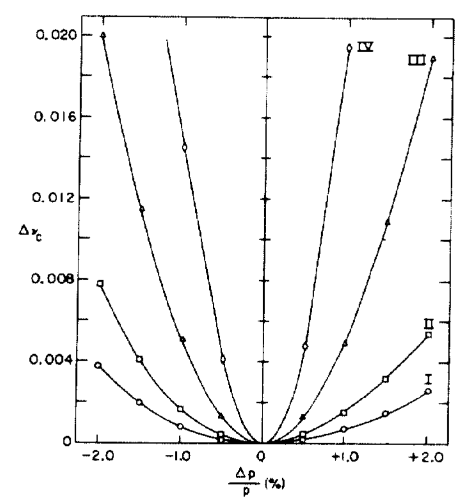

Fig. 3. ν_c vs $\Delta p/p$. Uncorrected chromaticities assume a lattice periodicity of 8.

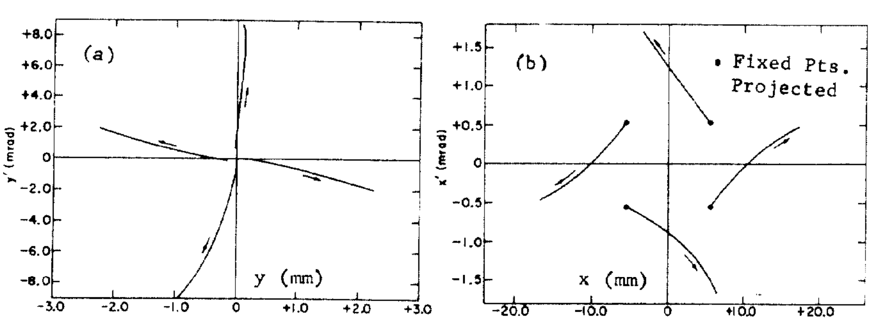

Fig. 5. 4th order sextupole resonance. Unstable trajectory. (a) Vertical projection, (b) Horizontal projection. Initial conditions [x=+5.5mm, x′=+0.55mrad, y=+0.25mm, y′=0.0].

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