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# A TWO-DIMENSIONAL INTENSE RELATIVISTIC BEAM EQUILIBRIUM*

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## Summary

The behavior of an intense relativistic charged particle beam is studied by determining a two-dimensional equilibrium solution of the laminar flow, monoenergetic particle dynamics equations. With these relations formulated in azimuthally symmetric cylindrical coordinates, the free parameter method is used to derive the most general similarity variable $\eta(r,z)$ appropriate to the equations and an exact solution is found in terms of this unknown. The solution is interpreted as a force-free converging or diverging particle beam propagating in the z direction within a conical drift chamber.

## Introduction

Experimental advances[1-3] of recent years have accelerated interest in transport phenomena for intense, relativistic charged particle beams. Propagation processes include injection into plasmas or neutral gases[4-7], vacuum propagation[8,9] and use of magnetic guide fields[10,11]. Only representative references have been listed in each of these areas.

While it is improbable that steady state conditions are closely approached in these pulsed beams, the nature of equilibria provides a useful tool in studying beam behavior. Hammer and Rostoker[5] have derived a one-dimensional equilibrium by assuming a phase space distribution function equal to a constant multiplier of Dirac delta functions in the invariants of classical mechanics. Benford and Book[6] found hollow beam equilibria in the presence of plasma back currents and the same authors have published a summary[12] of both axial and azimuthal intense relativistic beam equilibria. We have recently derived a one-dimensional vacuum equilibrium[9].

In this report, similarity analysis of the laminar flow, monoenergetic particle dynamics is used to develop a two-dimensional force-free particle flow, in contrast to the periodically-pinching, axially dependent equilibrium of Poukey, et al[13].

## Derivation of the Similarity Variable

The laminar equations of intense, relativistic charged particle flow[5] are coupled, nonlinear partial differential relations, describing self-consistent electromagnetic fields by Maxwell's equations and relativistic particle dynamics with the Vlasov equation and the Lorentz force law. Under conditions of azimuthal symmetry and time independence and an ad hoc assumption that the azimuthal component of velocity and axial component of magnetic induction are zero, these equations can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\gamma}{\partial r}\right]+\frac{\partial^2\gamma}{\partial z^2}=N \quad , \quad (1)$$

$$\frac{\partial b_\theta}{\partial z}-N\beta_r=0 \quad , \quad (2)$$

$$\frac{1}{r}\frac{\partial[rb_\theta]}{\partial r}+N\beta_z=0 \quad , \quad (3)$$

$$b_\theta=\frac{\partial[\gamma\beta_r]}{\partial z}-\frac{\partial[\gamma\beta_z]}{\partial r} \quad , \quad (4)$$

$$\gamma=(1-\beta_r^2-\beta_z^2)^{-1/2} \quad . \quad (5)$$

Eqs. (1) through (5) have been recast into dimensionless form by the dependent variable transformation

$$mc^2\,[\gamma^2(r,z)-1]+q\phi(r,z)=K_0 \quad , \quad (6)$$

$$N\,(r,z)=\frac{qL^2}{mc^2\epsilon_0}\,\rho(r,z) \quad , \quad (7)$$

$$\vec{b}\,(r,z)=-\frac{qL}{mc}\vec{B}\,(r,z) \quad , \quad (8)$$

$$\vec{\beta}\,(r,z)=\frac{\vec{v}\,(r,z)}{c} \quad . \quad (9)$$

$\phi(r,z)$ is electromagnetic scalar potential, $\rho(r,z)$ is charge density, $\vec{B}(r,z)$ is the magnetic induction vector and $\vec{v}(r,z)$ is the particle velocity. Constants m and q represent particle mass and charge, c is vacuum velocity of light and $\epsilon_0$ is the permittivity of free space. L is an arbitrary scale of distance.

Here we assume that dimensionless velocity $\vec{\beta}$, and consequently $\gamma$, can be expressed as a function of similarity variable $\eta(r,z)$. Restraints on the similarity variable are derived by replacing partial derivatives with respect to r and z by the appropriate chain rule operators. If Eqs. (1) through (5) are to be consistent with the similarity assumption

$$g(\eta)\frac{\partial\eta}{\partial r}+\frac{\partial\eta}{\partial z}\equiv 0 \quad , \quad (10)$$

where $g(\eta)$ is an arbitrary function. By comparison to the directional derivative along a curve of constant $\eta$ in the (r,z) plane, one concludes the most general form of the similarity variable to be

$$\eta\,(r,z)=\frac{r}{z-z_0} \quad (11)$$

where $z_0$ is an undefined constant.

## Similarity Solution

Given the explicit form of the similarity variable in Eq. (11), the chain rule operators transform the beam equations (1) through (4) into a set of ordinary differential equations,

$$\eta^2(\eta^2+1)\frac{d^2\gamma}{d\eta^2}+\eta(2\eta^2+1)\frac{d\gamma}{d\eta}=r^2N \quad , \quad (12)$$

$$\eta^2\frac{d\,[rb_\theta]}{d\eta}+r^2N\beta_r=0 \quad , \quad (13)$$

$$\eta\frac{d\,[rb_\theta]}{d\eta}+r^2N\beta_z=0 \quad , \quad (14)$$

$$rb_\theta + \eta^2 \frac{d[\gamma\beta_r]}{d\eta} + \eta \frac{d[\gamma\beta_z]}{d\eta} = 0 \qquad . \ (15)$$

After eliminating $r^2N$, $rb_\theta$ and components of $\overline{\beta}$, the system is reduced to a single differential equation in one dependent variable,

$$\frac{d^2\gamma}{d\eta^2} + \frac{2\eta^2+1}{\eta(\eta^2+1)}\frac{d\gamma}{d\eta} = \frac{\gamma}{\gamma^2-1}\left(\frac{d\gamma}{d\eta}\right)^2 \qquad . \ (16)$$

Fortuitous selection of a dependent variable transformations permitted an exact solution of Eq. (16) to be obtained.

$$\gamma = \cosh\,[\Lambda_1 \ln(\Lambda_2\psi)] \qquad , \ (17)$$

where $\Lambda_1$ and $\Lambda_2$ are constants of integration,

$$\psi = \frac{r}{[z-z_0] + [r^2 + (z-z_0)^2]^{1/2}} \, ,$$

$$= \sin\chi\,[1 + \cos\chi]^{-1} \qquad , \ (18)$$

and $\chi$ equals $\tan^{-1}(r/z-z_0)$.

## Physical Interpretation of the Solution

Before attempting to interpret the solution, we express the physical transport variables of the problem in terms of $\gamma$ and the independent variables. Including Eq. (6), these are written as

$$\rho(r,z) = \frac{mc^2\varepsilon_0\Lambda_1}{qr^2}\,\gamma\,(r,z) \qquad , \ (19)$$

$$B_\theta(r,z) = \frac{mc\Lambda_1}{qr}\,\gamma\,(r,z) \qquad , \ (20)$$

$$\vec{v}\,(r,z) = \frac{c[\gamma^2-1]^{1/2}}{\gamma[r^2 + (z-z_0)^2]^{1/2}}\,[r\vec{a}_r + (z-z_0)\vec{a}_z],$$

$$= \frac{c[\gamma^2-1]^{1/2}}{\gamma}\,[\sin\chi\vec{a}_r + \cos\chi\vec{a}_z] \qquad . \ (21)$$

One observes that Eq. (21) describes particles moving at constant velocity along straight trajectories on converging or diverging rays, emanating from the focal point $(0,\ \theta,\ z_0)$. $\chi$ represents the angle between a given trajectory and the z axis. To prevent infinite values of $\rho$ and $B_\theta$ as $r \to 0$, a centerline conical conductor with $\chi = \chi_{CL}$ is established. When a second (zero potential) conducting surface of revolution with $\chi = \chi_0$ is assumed, a force-free beam[14] in a conical drift chamber is described. This arrangement is illustrated in Figure 1.

The constants of integration can now be identified, using energy conservation Eq. (6) and Eq. (17) evaluated at $\gamma_0$ and $\gamma_{CL}$, corresponding to potential $\phi_0$ on the outer conical surface and $\phi_{CL}$ on the centerline electrode. In particular,

$$\Lambda_1 \ln\left[\frac{\sin\chi_{CL}\ (1+\cos\chi_0)}{\sin\chi_0\ (1+\cos\chi_{CL})}\right] = \ln\left[\frac{\gamma_{CL} + (\gamma_{CL}^2-1)^{1/2}}{\gamma_0 + (\gamma_0^2-1)^{1/2}}\right] \qquad . \ (22)$$

A current

$$I_{CL} = \frac{2\pi mc\Lambda_1}{\mu_0 q}\,\gamma_{CL} \qquad (23)$$

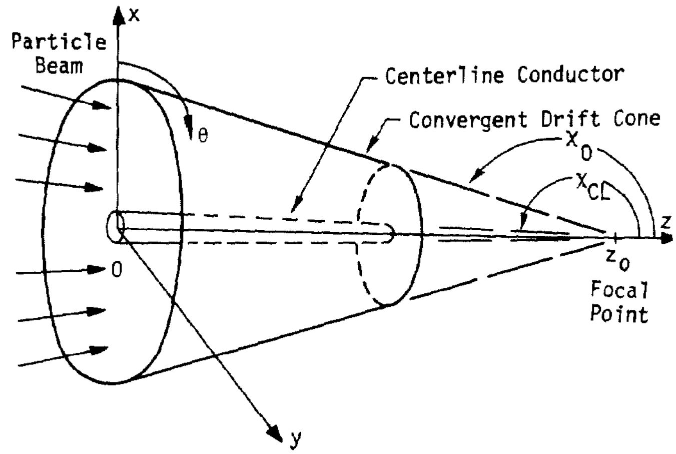


Figure 1. Geometric Interpretation of the Similarity Solution.

must be supplied through the center conductor, to provide the required discontinuity in $B_\theta$. Similar application of Ampere's Law at the outer boundary defines the total beam current to be

$$I_b = -I_{CL}\left[1 - \frac{\gamma_0}{\gamma_{CL}}\right] \qquad , \ (24)$$

a function depending only upon $\gamma_0$, $\gamma_{CL}$, and the geometric factor in $\Lambda_1$.

To illustrate the character of the solution, Figures 2 and 3 show the current propagated in a 10.5 MeV electron beam and required centerline current, as a function of positive and negative centerline potentials, respectively. For these plots, the geometric factor with $\chi_0$ = 150° and $\chi_{CL}$ = 179.6° is used.

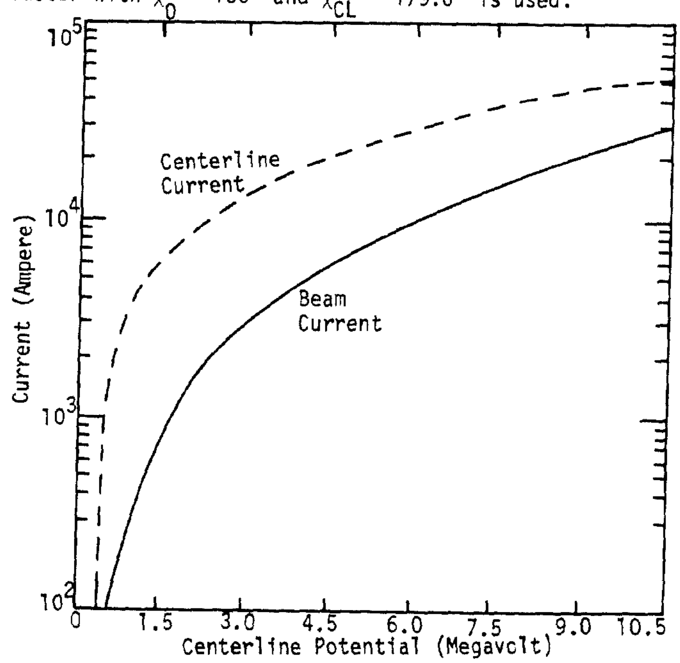


Figure 2. Currents at Positive Centerline Potentials

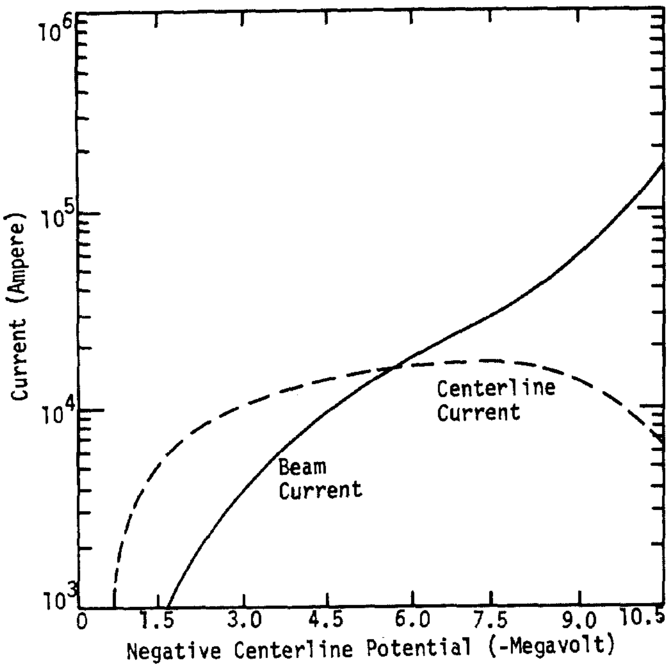


Figure 3. Currents at Negative Centerline Potentials

For increasingly positive $\phi_{CL}$, the magnitudes of $A_1$ and $\gamma_{CL}$ grow monotonically. The centerline current, in this case positive and generated at least in part from the beam return current, and $I_b$ are increasing and their ratio approaching unity.

At negative centerline potentials $I_{CL}$ is negative, in the same sense as beam current, and must be externally supplied. For values of $\phi_{CL}$ such that $\gamma_{CL} < \gamma_0/2$, beam current exceeds $I_{CL}$ and has magnitude similar to the outputs of present-day pulsed accelerators[2,3] in this voltage range.

In the limit of small convergence angles, the similarity solution reduces to the one-dimensional vacuum beam equilibrium of Reference 9.

## Conclusions

Similarity solution of the laminar, monenergetic dynamic equations at relativistic particle energies have demonstrated theoretical existence of a force-free two-dimensional charged particle beam. The result corresponds to a converging or diverging beam in a conical drift chamber, with current magnitudes of practical interest.

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