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RECTILINEAR TRANSITION FLOW OF INTENSE CHARGED PARTICLE BEAMS*

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Summary

Charged particle flow in planar geometry is generally classified into two phenomenological regimes. At relatively low current injection densities, less than a critical function of length and potential across the drift space and entrance kinetic energy, all charges propagate across the entire region and the flow is termed injection limited. Space charge limited flow occurs at supercritical injection densities and is characterized by partial particle reflection at the plane of minimum kinetic energy. Solutions of the laminar, monoenergetic beam equations demonstrate the existence of a transition region, rather than a single critical current density, in which either flow may exist.

Introduction

Charged particle propagation between planar electrodes has been extensively considered, since Child[1] reported studies of space charge limiting in diode regions. Amboss[2] has extended the analysis to thermally distributed particles. The diode equations have been solved at relativistic energies by Acton[3], Boers and Kelleher[4] and Jory and Trivelpiece[5]. Poukey and Rostoker[6] investigated injection of relativistic particles into a one-dimensional vacuum drift space.

The geometry of rectilinear charged particle flow is illustrated in Figure 1. Drift space is a vacuum region, enclosed by two unbounded and infinitely conducting planes separated by a distance L. A single specie charged particle beam, assumed to be nonthermal and monoenergetic, is normally and uniformly injected with entrance kinetic energy K_0 at the reference (zero) potential surface at x = 0. The injection current density is denoted by J_0.

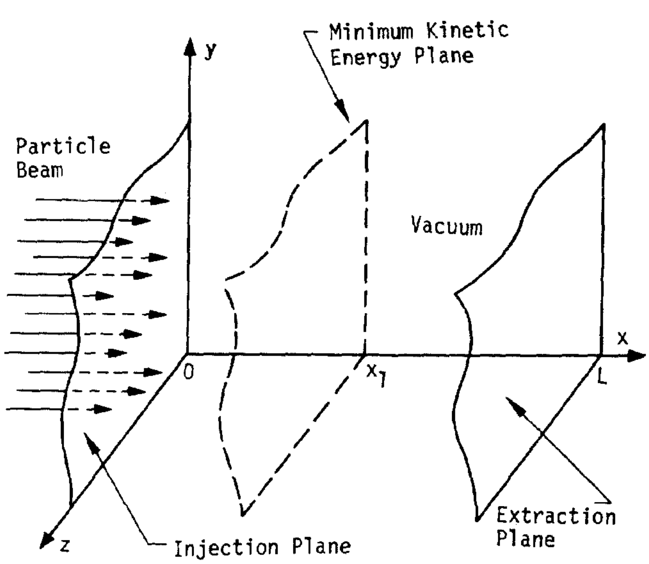

Figure 1. Planar Drift Space

Initially, the space charge effect retards particle motion. At $x = x_1$, minimum kinetic energy is achieved and some particles may be reflected toward the injection surface. To the right of the plane of minimum kinetic energy, the self-consistent electric field is directed to accelerate charges toward the extraction surface.

The problem is clearly nonphysical for several reasons, but represents an approximation to finite thermal beams, which propagate in a short drift zone compared to the radial beam dimension and is useful in studying general beam behavior.

Beam Equations and Solutions

Here, we consider the steady state solutions of the laminar beam equations, rather than the transient response of reference 6. Laminar charged particle flow is governed by Maxwell's equations, relating electromagnetic fields and sources, the Vlasov formulation of particle distribution function evolution and the Lorentz force law[7]. After additional simplifications due to the idealized geometry and time independence, the governing equations are written as

$$\nabla^2\phi(x) = \frac{d^2\phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0} \quad , \quad (1)$$

$$K_0 = mc^2\,[\gamma(x)-1] + q\phi(x) \quad , \quad (2)$$

$$TJ_0 = \frac{c\rho(x)T[\gamma^2(x)-1]^{1/2}}{[2-T]\gamma(x)} \; , \; x \le x_1 \quad ,$$

$$= \frac{c\rho(x)[\gamma^2(x)-1]^{1/2}}{\gamma(x)} \; ; \; x \ge x_1 \quad . \quad (3)$$

In this formulation, $\phi(x)$ is the electromagnetic scalar potential, $\rho(x)$ is the charge density, ε_0 equals permittivity of free space, m and q represent particle mass and charge and $\gamma(x)$ is defined by $(1-v^2(x)/c^2)^{-1/2}$, where v(x) is particle speed. T symbolizes the fraction of injection current transmitted into the drift region to the right of the minimum kinetic energy plane.

Two variables are eliminated to produce the second order, nonlinear differential equation

$$\frac{d^2\gamma}{dx^2} = \Lambda_1 \frac{\gamma}{[\gamma^2-1]^{1/2}} \quad , \quad x \le x_1 \quad ,$$

$$= \Lambda_2 \frac{\gamma}{[\gamma^2-1]^{1/2}} \quad , \quad x \ge x_1 \quad , \quad (4)$$

where

$$\Lambda_1 = \frac{[2-T]qJ_0}{mc^3\varepsilon_0} \quad , \quad (5)$$

$$\Lambda_2 = \frac{TqJ_0}{mc^3\varepsilon_0} \quad . \quad (6)$$

Eq. (4) is solved subject to the boundary conditions

(a) $\gamma(x)$ is continuous at $x = x_1$,

(b) $d\gamma/dx = 0$ at $x = x_1$,

(c) $\gamma(0)$ and $\gamma(L)$ are known from Eq. (2).

The first integral of Eq. (4) is elementary, yielding

$$\frac{d\gamma}{dx} = -[2\Lambda_1]^{1/2}\left\{\left[\gamma^2(x)-1\right]^{1/2} - \left[\gamma^2(x_1)-1\right]^{1/2}\right\}^{1/2}, \quad x \leq x_1 ,$$

$$= [2\Lambda_2]^{1/2}\left\{\left[\gamma^2(x)-1\right]^{1/2} - \left[\gamma^2(x_1)-1\right]^{1/2}\right\}^{1/2}, \quad x \geq x_1 . \quad (7)$$

The second integral can be written exactly in elliptic and Zeta functions[5,8] or solved in series form[3,4]. However, numerical integration is simple and direct. The nature of the transition region is equally well illustrated by the nonrelativistic approximation, which can be written in simple closed form. For K_0 and $|q\phi(L)|$ much less than mc^2,

$$x-x_1 = -\left[\frac{2m\varepsilon_0}{9(2-T)qJ_0}\right]^{1/2}\left[v(x)+2v(x_1)\right]\left[v(x)-v(x_1)\right]^{1/2}, \quad x \leq x_1 ,$$

$$= \left[\frac{2m\varepsilon_0}{9TqJ_0}\right]^{1/2}\left[v(x)+2v(x_1)\right]\left[v(x)-v(x_1)\right]^{1/2}, \quad x \geq x_1 . \quad (8)$$

Injection Limited Flow

By definition, injection limited flow implies that every charged particle propagates across the entire length of drift space and the transmission fraction is necessarily unity. Application of boundary conditions at $x = 0$ and $x = L$ are sufficient to determine the unknown coordinate of kinetic energy minimum (x_1) and the particle velocity at this plane, $\gamma(x_1)$ or $v(x_1)$.

The maximum current density of the injection limited regime can be determined by solving Eq. (8) for J_0 and evaluating extrema by differentiation with respect to $v(x_1)$. The value corresponding to maximum J_0 is a solution of the equation

$$[v(0)-2v(x_1)][v(L)-v(x_1)]^{1/2} + [v(L)-2v(x_1)][v(0)-v(x_1)]^{1/2} = 0 , \quad (9)$$

the only physically acceptable root of which is

$$v(x_1) = \frac{v(0)v(L)}{v(0)+v(L)} . \quad (10)$$

Substitution of this result into Eq. (8) identifies the desired result

$$\left|J_{0_{max}}\right| = \left|\frac{2m\varepsilon_0}{9qL^2}\right| [v(0)+v(L)]^3 . \quad (11)$$

At injection current densities exceeding this value, no physical solution exist for $T = 1$.

It can further be shown, that for injection current densities in excess of

$$\left|J_0\right| = \left|\frac{2m\varepsilon_0}{9qL^2}\right| [v^3(0)+v^3(L)] , \quad (12)$$

the transport variables ρ, ϕ and γ or v do not possess unique solutions.

To illustrate these properties of injection limited charge flow, Figure 2 graphs the propagation length achieved by an electron beam of given injection current density and entrance kinetic energy of 5 MeV as a function of $\gamma(x_1)$. Dual solutions exist for minimum values of the relativistic parameter in the interval from unity to 4.51. Maximum propagation length corresponds to $\gamma(x_1)$ equals 1.71.

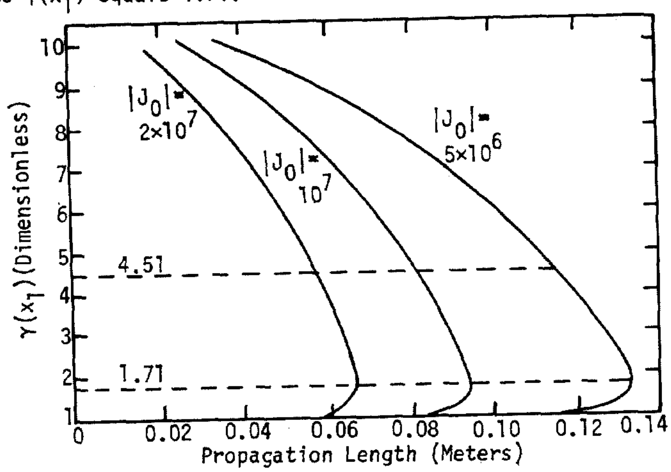

Figure 2. Properties of Injection Limited Electron Beam Propagation.

Space Charge Limited Flow

In the case of space charge limited flow, particle reflection at the plane of minimum kinetic energy requires that $\phi(x_1)$ equals K_0/q and $\gamma(x_1)$ is unity. Boundary conditions are, therefore, determinants for the values x_1 and T.

We define the critical injection density to be the value of J_0 at the onset of space charge limiting, that is, the density at which both $\gamma(x_1)$ and the transmission coefficient T are unity. For the nonrelativistic approximation, Eq. (8) demonstrates the critical value to be

$$\left|J_{0_{crit}}\right| = \left|\frac{2m\varepsilon_0}{9qL^2}\right| [v^3(0)+v^3(L)] . \quad (13)$$

Therefore, space charge limited solutions of the beam equations exist at current densities well below the maximum injection limited value, given by Eq. (11). Furthermore, since Eqs. (12) and (13) are identical, one notes that the onset of space charge limiting and the lower bound of doublevalued injection limited solutions correspond exactly.

Conclusions

Solutions of the laminar, monoenergetic, steady state beam equations have shown the existence of a region of injection current densities in which either space charge limited or injection limited charged particle flow may theoretically exist. In this same band, the injection limited solution has a multiplicity of two. In the nonrelativistic approximation, the upper and lower extremes of the region have been exactly defined,

$$[v^3(0)+v^3(L)] \leq \frac{9qL^2J_0}{2m\varepsilon_0} \leq [v(0)+v(L)]^3 . \quad (14)$$

Numerical evaluation of transport solutions at relativistic energies demonstrates a completely analogous situation. These findings are summarized in Figure 3 for electron beams of two entrance kinetic energies. The broken curves of transmitted current density versus injected particle density in a 10-centimeter drift space illustrate the injection limited solutions for the problem, while solid curves represent space charge limited flows. The overlapping region is clearly seen.

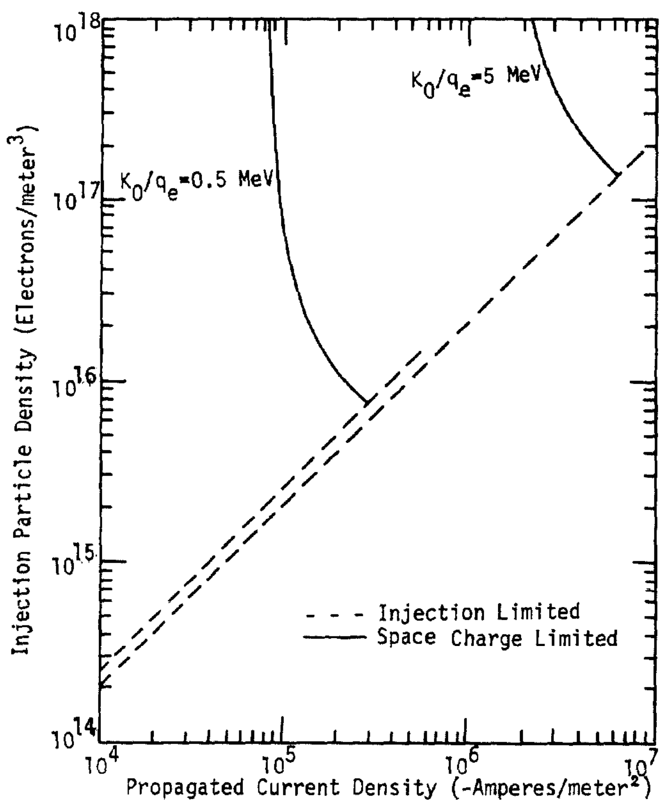

Figure 3. Regimes of Intense, Charged Particle Flow

The duality of charged particle flows in the transition region is analgous to that in fluid dynamics. Above a critical Reynold's number, either laminar or turbulent flow may occur at identical entrance and boundary conditions.

While stability analysis has not been performed, it is anticipated that supercritical injection limited propagation is unstable. As in the fluid flow example, in the presence of perturbations, catastrophic change to a space charge limited condition may be expected.

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