SPACE CHARGE DEFOCUSSING OF BUNCHED BEAMS

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The discussion of the force between moving charges is a curious and interesting problem, since the solution is not as obvious and straightforward as one might suppose, because ultimately an electrodynamic theory must be based upon experience. The electromagnetic theory, for example, is too broad a theory to be verifiable by any single experiment; it is the product of several singular observations. However the range of observations upon which the theory rests is extremely limited (as regards particle energy, for example) and it cannot be expected that it would not be modified to account for new observations.

The electromagnetic theory is probably the most satisfactory of the various physical theories which we presently possess, in spite of its incompleteness and inconsistencies, and it is therefore on this basis that we propose to estimate the solution of the problem of dispersion due to space charge. In such a case, it is possible to solve an electrodynamic problem in the stationary (or laboratory) frame using the electromagnetic fields in that frame. It is, alternately, possible to solve an electrodynamic problem in a moving frame, using electrokinetic forces (no magnetic field) and to translate the observations into the laboratory frame using the Lorentz transform. The latter is sometimes preferred because one can usually employ non-relativistic (linear) mechanics and, in addition, magnetic forces due to ion motion in the moving frame are negligible.

Continuous Beams

Several analysts have obtained non-relativistic solutions of the paraxial ray equation for a solid, constant velocity beam in free space. McGregor-Morris and Mines[1] calculated beam spreading of an initially parallel beam due to self-repulsion of the electrons. Watson[2] extended this treatment by including the magnetic forces involved. The subject having been broached, and because of its technical importance, extensive treatments and discussion were given by Fowler and Gibson[3]; Van Boories and Dosse[4]; Thompson and Headrick[5]; Spangenberg, Field and Helm[6]; and Schwartz[7]. The additional effect of thermal velocity limitations (on relatively low voltage beams) has been investigated by Cutler and Hines[8]; Hollway[9]; and Danielson, Rosenfeld and Saloom[10]. Further, some effects of the ionization of residual gas thus neutralizing space charge forces has been examined by Hernqvist and Linder[11].

The relativistic solution of beam spreading due to space charge forces in free space has been discussed by Yadavalli[12]. Numerous schemes to foil space charge forces have been invented which are outside the scope of this particular discussion.

The method of analyzing beam spreading, applicable to low voltage beams with little spreading is outlined here for comparison to the relativistic analysis.

Expressing the non-relativistic Lorentz force in cylindrical coordinates (using dots to indicate a derivative with respect to time)

$$\ddot{z} = \frac{e}{m_0}\left(E_z + \dot{r} B_\varphi - r\dot{\varphi} B_r\right) \tag{1a}$$

$$\ddot{r} - r\dot{\varphi}^2 = \frac{e}{m_0}\left(E_r + r\dot{\varphi} B_z - \dot{z} B_\varphi\right) \tag{1b}$$

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = \frac{e}{m_0}\left(E_\varphi + \dot{z} B_r - \dot{r} B_z\right) \tag{1c}$$

By symmetry E_z, E_φ, B_Z and B_r vanish so that these equations become

$$\ddot{z} = 0 \tag{2a}$$

$$\ddot{r} - r\dot{\varphi}^2 = \frac{e}{m_0}\left(E_r - \dot{z} B_\varphi\right) \tag{2b}$$

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0 \tag{2c}$$

Note that (where primes indicate differentation with respect to the Z-axis)

$$\dot{r} = r'\dot{z}$$
$$\ddot{r} = r''\dot{z}^2 + r'\ddot{z}$$

and that we may put (the paraxial ray assumption, $r\dot{\varphi} < \dot{r} < \dot{Z}$)

$$\dot{r}^2 + (r\dot{\varphi})^2 + \dot{z}^2 \doteq \dot{z}^2 = 2\frac{e}{m_0}V$$

or

$$\ddot{z} = \frac{e}{m_0}V'$$

Thus, Eq (2b) may be put in the form

$$2Vr'' + V'r' = E_r - \dot{z}B_\varphi \tag{3}$$

For a constant potential beam $V' = 0$; ignoring B_φ at low voltages and using Gauss' theorem to obtain the radial electric field from the total charge,

$$E_r = \frac{I}{2\pi\epsilon_0 \dot{z} r}$$

we have, finally,

$$r'' = \frac{I}{4\epsilon_0\sqrt{\frac{2e}{m_0}}V^3\pi r} \tag{4}$$

Integrating once, with the boundary condition $r' = r'_0$, $r = r_0$, $z = 0$

$$(r')^2 - (r'_0)^2 = \frac{I}{\pi 2\epsilon_0\sqrt{\frac{2e}{m_0}V^3}\,r_0}\ln\frac{r}{r_0} \tag{5}$$

Reintegrating,

$$\sqrt{\frac{I}{2\pi\epsilon_0\sqrt{\frac{2eV^3}{m_0}}}}\ \frac{z}{r_0} = 2e^{-\left(\frac{r_0'}{r_0}\right)^2}\int_{r_0'/r_0}^{\sqrt{\ln\frac{r}{r_0}+\left(\frac{r_0'}{r_0}\right)^2}} e^{x^2}dx, \quad x=\frac{r_0'}{r_0} \tag{6}$$

This integral has been tabulated in numerous places,

The dispersion of a drifting beam due to space charge is considerably reduced at higher energies as compared to a non-relativistic beam, not only because of the effective increase in mass of the particles at higher energies, but also because of the reduction of the self-repulsive force.

Unfortunately there is a fundamental difficulty in obtaining a precise evaluation of the beam dispersion. The radial equation of motion for an electron of velocity v at a distance r from the axis of the beam is of the form

$$\frac{d}{dt}\left(\gamma\frac{dr}{dt}\right) = \frac{eI}{m_0 v}\left(1-\frac{v^2}{c^2}\right)\Omega$$

where Ω is a function of the distribution of electrons over the beam cross-section and the initial displacement of the electron whose motion is required. If the beam actually disperses it is impossible to know this distribution until the equations of motion are solved, even if we assume an initial distribution. However, the general characteristics of the forces involved are evident, and if we make the assumptions that the particle energy is not affected by the dispersion and that the distribution is uniform over the beam cross-section, we can estimate the dispersion of a beam of particles of initial diameter $2r_o$ whose motions are also initially parallel to the beam axis.

The force on a particle at a specified radius only depends on the charge inside that radius. If the charge density of the beam is ρ and the total current I

$$I = \rho v\pi r_0^2$$

By Gauss' theorem the outwardly directed force of the radial electric field due to the beam charge is, at a radius r, given by

$$E = \frac{\rho}{2\pi\epsilon_0 r} = \frac{Ir}{2\pi\epsilon_0 v r_0^2}$$

In addition the beam current has a magnetic field associated with it, producing an inwardly directed force, and this is given by

$$B = \mu_0 H = \mu_0\frac{I}{2\pi r}\left(\frac{r}{r_0}\right)^2$$

The total radial force on a particle a distance r from the axis of the beam is thus

$$eE + evB = \frac{eIr}{2\pi r_0^2}\left(\frac{1}{\epsilon_0 v} - \mu_0 v\right) = \frac{eI\eta r}{2\pi r_0^2}\ \frac{1-\beta^2}{\beta}$$

where β = v/c and $\eta = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.

The radial motion of the particle is therefore given by

$$\frac{d^2r}{dt^2} = \frac{eIr\eta}{\gamma m_0 2\pi r_0^2}\ \frac{1-\beta^2}{\beta} \tag{7}$$

The solution of this equation for the beam envelope ($r = r_o$) is straightforward. Integrating, with the boundary conditions $dr_o/dt = 0$, $r_o = a$, $t = 0$ we have

$$\frac{dr_0}{dt} = \sqrt{2q\ln\frac{r_0}{a}} \tag{8}$$

where $q = eI\eta\,(1-\beta^2)/\gamma m_o 2\pi\beta$. This expression cannot be integrated in terms of elementary functions; however, a substitution of variable will result in form which may be conveniently tabulated. Let

$$x = \ln\frac{r_0}{a}$$

Then we may write the solution of Eq (8) in the form

$$\frac{\sqrt{2q}\,t}{a} = \int_0^x \frac{e^x}{\sqrt{x}}\,dx = 2\sum_0 \frac{x^{\frac{2n+1}{2}}}{n!(2n+1)} \tag{9}$$

where $t = \frac{z}{c}\frac{\gamma}{\gamma^2-1}$ is the distance at which the beam envelope has the diameter r_o/a.

With this solution it is possible to solve the original differential equation. Rewriting Eq (7) in the form

$$\frac{d^2r}{dt^2} = \frac{qr}{r_0^2}$$

we observe that the solution is

$$\frac{r}{b} = \frac{r_0}{a}$$

where b is the particle radius at t = O, and this is easily proved by noting that

$$\frac{d^2r}{dt^2} = \frac{b}{a}\ \frac{d^2r_0}{dt^2} = \left(\frac{r}{r_0}\right)\left(\frac{q}{r_0}\right) \tag{10}$$

It is therefore evident that the assumption of uniform charge density in the beam cross-section is consistent with the solution.

Note that the particles in the beam move as if they were under the influence of repulsive force whose intensity varies inversely as the distance of the particle from the center of repulsion, that is the beam axis. The radial velocity of the particles is acquired at the expense of the potential energy of the beam; the axial velocity remains constant. It should be noted also that the above solution is a free-space analysis; in an enclosed space, such as a conduction beam tube, the space charge forces would be reduced.

The above solution for the beam expansion due to space charge forces was treated in the laboratory frame using electromagnetic field equations. It is also possible to solve this problem, as mentioned earlier, electrodynamically in a frame moving with the beam, using a relativistic transformation to transfer the solution into the laboratory frame.

In the moving frame the divergence equation is valid

$$\nabla \cdot \vec{D} = \rho$$

but the estimate of the charge density must be that of the moving observer. The laboratory frame estimate ρ_1 is related to the moving frame estimate ρ by the condition $\rho_1 = \gamma\rho$ since the moving frame 'sees' the beam dimensions 'contracted' in the direction of motion. (And thus, the question of the validity of Gauss' equation when the charges are in motion is evident, $\nabla \cdot \vec{D} = \rho_1/\gamma$ where the charge density is based upon a determination in the laboratory frame[13].)

The divergence equation may then be solved, very approximately, since

$$\frac{\partial}{\partial r}(rE_r) = \frac{\rho_1 r}{\gamma \epsilon_0}$$

and hence,

$$E_r = \frac{\rho_1 r}{2\gamma\epsilon_0}$$

with the boundary conditions $E_r = 0$, $r = 0$.

The current flow (only observed in the laboratory system) may be substituted for the beam density since

$$I = \rho_1 v \pi r^2$$

The expansion of the beam may be calculated in the moving frame, using the equation of motion for a peripheral particle,

$$\frac{d^2r}{dt^2} = \frac{e}{m_0} E_r = \frac{eI}{m_0 2\pi\epsilon_0 r v \gamma}$$

Integrating, with the boundary conditions $dr/dt = 0$, $r = r_0$ and noting that $\epsilon_0 C = 1/\eta$ and $\beta = v/c = \sqrt{\gamma^2-1}/\gamma$

$$\left(\frac{dr}{dt}\right)^2 = \frac{eI\eta}{m_0\pi\sqrt{\gamma^2-1}} \ln\frac{r}{r_0}$$

This expression cannot, as above, be integrated again conveniently. But, letting $X = \ln(r/r_0)$ the above expression can be put in the form

$$\sqrt{\frac{eI\eta}{m_0\pi\sqrt{\gamma^2-1}}}\ \frac{t}{r_0} = \int \frac{e^x}{\sqrt{x}}\,dx$$

The time in this expression may be converted into laboratory time since $t_1 = \gamma t$; however, if we wish to express the beam expansion in terms of drift distance in the laboratory frame,

$$t_1 = \frac{z}{v} = \frac{z}{c}\ \frac{\gamma}{\gamma^2-1}$$

and the final expression becomes

$$\sqrt{\frac{eI\eta}{m_0 c^2\pi(\gamma^2-1)^{3/2}}}\ \frac{z}{r_0} = \int \frac{e^x}{\sqrt{x}}\,dx \tag{11}$$

which is precisely the same as that derived above.

Bunched Beams

To calculate the expansion of an ion bunch in free space due to space charge forces it is convenient to solve the problem in a frame moving with the bunch (assumed to be composed initially of mono-energetic particles in the laboratory frame), and then transform the results into the laboratory frame[14].

We assume for simplicity a spherical packet of N electrons travelling at a velocity $v = \beta c$. (This packet will not be spherical in the laboratory frame.) Then, for a spherically symmetric distribution a particle at the edge of the packet experiences a radial force, and its motion is described by

$$\frac{d^2r}{dt^2} = \frac{Ne^2}{4\pi\epsilon_0 m_0 r^2} \tag{12}$$

Integrating once,

$$\left(\frac{dr}{dt}\right)^2 = \frac{Ne^2}{2\pi\epsilon_0 m_0}\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{13}$$

where $dr/dt = 0$, $r = r_0$, $t = 0$ are boundary conditions. Integrating again,

$$\sqrt{\left(\frac{r}{r_0}\right)^2 - \frac{r}{r_0}} + \operatorname{arccosh}\sqrt{\frac{r}{r_0}} = \sqrt{\frac{Ne^2}{2\pi\epsilon_0 m_0}}\ \frac{t}{r_0^{3/2}} \tag{14}$$

Using the Lorentz transform, we can now convert to the laboratory frame:

$$t_1 = \gamma t \qquad r_1^2 = r^2(1-\beta^2\cos^2\vartheta) \tag{15}$$

where ϑ is the angle between r and the direction of motion measured in the moving frame. It is easily shown that the angle ϑ_1 in the laboratory frame is related to that in the moving frame by the expression (where V is the relative velocity of the two frames) by[15]

$$\tan\vartheta = \frac{\sin\vartheta_1}{\gamma\left(\cos\vartheta_1 + \frac{V}{u_1}\right)} \tag{16}$$

where u_1 is the particle velocity in the laboratory frame and related to that in the moving frame, u, by

$$u = u_1 \frac{\sqrt{1 + 2\frac{v}{u_1}\cos\vartheta_1 + \left(\frac{v}{u_1}\right)^2 - \left(\frac{v}{c}\sin\vartheta_1\right)^2}}{1 + \frac{u_1 v}{c^2}\cos\vartheta_1} \tag{17}$$

Inserting these transformations in the above equations, where we have also put the drift distance in place of the time, $t_1 = Z/v = \frac{Z}{c}\frac{\gamma}{\sqrt{\gamma^2-1}}$ we have: (1) for the transverse expansion,

$$\sqrt{\left(\frac{r}{r_{ot}}\right)^2 - \left(\frac{r}{r_{ot}}\right)} + \operatorname{arccosh}\sqrt{\frac{r}{r_{ot}}} = \sqrt{\frac{Ne^2}{c^2 2\pi\epsilon_o m_o r_{ot}(\gamma^2-1)}}\,\frac{Z}{r_{ot}} \tag{18}$$

and (2) for the longitudinal expansion,

$$\sqrt{\left(\frac{r}{r_{o\ell}}\right)^2 - \frac{r}{r_{o\ell}}} + \operatorname{arccosh}\sqrt{\frac{r}{r_{o\ell}}} = \sqrt{\frac{Ne^2}{2\pi\epsilon_o m_o c^2 r_{o\ell}\gamma^3(\gamma^2-1)}}\,\frac{Z}{r_{o\ell}} \tag{19}$$

There are several objections to the above solution. First, it is not unreasonable that the number of electrons in the bunch is sufficiently large, such that the field intensity at the periphery of the bunch

$$E_r = \frac{Ne}{4\pi\epsilon_o r^2} \tag{20}$$

is also large enough to require the use of relativistic equations of motion in the moving frame (the frame stationary with respect to the bunch). Secondly, there is no single transformation to return to the laboratory frame, since the energy of each particle will depend upon its coordinates.

The first comment introduces considerable complication and this is customarily avoided in such problems by using non-relativistic mechanics in the moving frame. For example, in this instance the equation of motion in the moving frame is, precisely, where a dot indicates differentiation with respect to time,

$$\frac{d}{dt}(\gamma\dot{r}) = \frac{Ne^2}{4\pi\epsilon_o m_o r^2} \tag{21}$$

but since

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{\dot{r}}{c}\right)^2}}$$

we may write Eq (21) in the form

$$\frac{\ddot{r}}{\left[1-\left(\frac{\dot{r}}{c}\right)^2\right]^{3/2}} = \frac{Ne^2}{4\pi\epsilon_o m_o r^2}$$

Integrating, with the boundary conditions $dr/dt = 0$, $r = r_o$, $t = 0$,

$$\frac{1}{\sqrt{1-\left(\frac{\dot{r}}{c}\right)^2}} = 1 - \frac{Ne^2}{4\pi\epsilon_o m_o c^2}\left(\frac{1}{r} - \frac{1}{r_o}\right)$$

Re-arranging, reintegrating and simplifying,

$$\left(1+\frac{a}{r_o}\right)\sqrt{\left[\frac{r}{a}\left(1-\frac{a}{r}+\frac{a}{r_o}\right)^2 + \left(1-\frac{a}{r}+\frac{a}{r_o}\right) - 1\right]^2 - 1}$$
$$+\cosh^{-1}\left[\frac{r}{a}\left(1-\frac{a}{r}+\frac{a}{r_o}\right) + \left(1-\frac{a}{r}+\frac{a}{r_o}\right) - 1\right] = \frac{ct}{a}\left[\frac{a}{r_o}\left(1+\frac{a}{r_o}\right)\right]^{3/2}$$

where $a = Ne^2/4\pi\epsilon_o m_o C^2$ for brevity. As remarked above, the transform into the laboratory system is coordinate dependent and this results in further complication, although the solution is straightforward.

Probably of more importance is the energy spread which occurs in the bunch due to its dispersion, and its effect on a beam transport system.

The transformation of the bunch into the laboratory frame of reference is actually much more complicated than superficially indicated above. What we are looking for is, presumably, the velocity of the coordinate system relative to which the momentum of the particle ensemble is equal to zero. For a single particle evidently this is

$$\vec{V} = \frac{\vec{p}\,c^2}{E}$$

where $\vec{p}$ and E are the momentum and energy of the particle. For an ensemble of particles if we use the total momentum and total energy (with respect to the laboratory frame), then since the Lorentz transform is linear,

$$\vec{V} = \frac{c^2\Sigma\vec{p}}{\Sigma E}$$

This quantity $\vec{V}$ does not have the form of a total derivative of any quantity with respect to time: for this reason the concept of center of mass for accelerated particles cannot be used in relativistic mechanics[16].

On the other hand, one might consider the rather tedious alternative of 'tracking' every particle of the ensemble and 'reconstituting' the bunch at any desired location.

The bunch is not a rigid body and, of course, the Lorentz transformation (for intervals) does not strictly apply. The original spherical packet, for example, has a somewhat cometary appearance after drifting some distance.

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