THROBBING BEAM INSTABILITIES IN PARTICLE ACCELERATORS AND STORAGE RINGS

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Summary

Finite conductivity of the vacuum chamber wall can cause unstable coherent transverse oscillations of the center of charge and also oscillations of the transverse cross section of a beam of charged particles. The former case, which can be characterized by a dipole oscillation, has been studied extensively by others.[1, 2] In this work, a study has been conducted of the cross sectional oscillations of a nearly circular beam centered in a circular pipe, and a self-consistent solution has been obtained for both monopole and quadrupole oscillations. Dispersion relations for the oscillation frequencies have been found, and conditions for stability have been deduced. For the quadrupole instability the growth rate and threshold are very close to those obtained for the dipole instability,[1, 2] differing only by a geometrical factor of order unity; whereas, for the monopole instability, the growth rates are so small that the oscillations in present electron accelerators and storage rings will be suppressed by radiation damping.

Introduction

Laslett, Neil and Sessler[1] have demonstrated the possibility for a beam of charged particles to have unstable coherent transverse oscillations due to the finite conductivity of the vacuum chamber walls. In their analysis, Laslett et al. assumed that the longitudinal charge density variation was zero; Courant and Sessler,[2] and Dikanskii and Skrinskii[3] found that it is also possible to have unstable coherent transverse oscillation for the case where the beam was bunched due to the wake field of one bunch acting upon another bunch and upon itself in successive revolutions. In the above papers the unstable oscillations were oscillations of the center of the beam. In this work we treat some cases of the oscillation of the transverse size of a longitudinally uniform or bunched beam which is confined within a circular pipe, with the center of the beam fixed along the pipe axis.

In Sec. II we characterize the monopole and quadrupole oscillations of a beam, uniform or bunched, by some assumed charge densities and present the equations of motion for the particles.

Section III contains the body of the analysis in which we solve the Vlasov equation combined with the equations of motion given in Sec. II for a self-consistent charge distribution and obtain a dispersion relation for the oscillation frequencies. The dispersion relation is analyzed in Sec. IV, culminating in the determination of the growth times and the stability criteria for the oscillations.

Some numerical illustrations of the results and experimental observations are given in Sec. V. The effects of Landau damping, resulting from a spread in the amplitude of oscillations, is considered in Sec. VI.

II. Monopole and Quadrupole Charge Oscillations

In this section we characterize by some simple models the monopole and quadrupole oscillations of a uniform or bunched beam inside a metallic vacuum chamber. As the major curvature of the vacuum chamber has little influence on the calculation of the fields, the chamber is taken to be a straight pipe of radius b. The particles in the beam are assumed to be moving longitudinally in the z-direction, along the axis of the pipe, with a constant velocity, v.

The unperturbed beam is taken as uniform in the transverse cross section over a

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circle of radius a, with the center of the beam fixed along the pipe axis. Thus, for the charge density of the unperturbed beam

$$\rho_0 = \frac{e\lambda}{\pi a^2} H(a - r) \ , \tag{2.1}$$

where H(x) is the Heaviside unit function. For the uniform beam $e\lambda$ is the charge per unit length, while for the bunched beam $e\lambda = eN\, f(z - z_0 - vt)$ with the function f(x) normalized such that eN is the charge in the bunch $\left(\int_{-\infty}^{\infty} f(x)\,dx = 1\right)$.

In the perturbed beam, we assume the radius varies as $(a + \xi)$ for monopole oscillation and $(a + \xi \cos 2\theta)$ for quadrupole oscillation with the perturbation amplitude ξ given by

$$\xi = \begin{cases} \xi_0 e^{i(kz - \omega t)} & (2.2a) \\ \xi_0 e^{-i\omega t} & (2.2b) \end{cases}$$

As a consequence of the perturbation, to first order in ξ the charge density can be written as

$$\rho = \rho_0 + \xi \rho_1 \ , \tag{2.3}$$

where

$$\rho_1 = \begin{cases} \frac{e\lambda}{\pi a^2}\left[\delta(a-r) - \frac{2}{a} H(a-r)\right] \ \text{(monopole beam)} & (2.4a) \\ \frac{e\lambda \cos 2\theta}{\pi a^2}\, \delta(a - r) \ \text{(quadrupole beam)} & (2.4b) \end{cases}$$

For a particle in the beam, the equations of motion are

$$\dot{p}_x = -(\omega_0^2 + \xi K)x \ , \quad \dot{x} = P_x \tag{2.5a}$$

$$\dot{p}_y = -(\omega_0^2 \pm \xi K)y \ , \quad \dot{y} = P_y \tag{2.5b}$$

$$\dot{p}_z = 0 \ , \quad \text{and} \quad \dot{z} = v \tag{2.5c}$$

where P_x, P_y and P_z are the conjugate momenta, and the upper and lower signs correspond to the monopole and quadrupole oscillations. The contribution of both the external fields and the electromagnetic fields due to ρ_0 are included in ω_0^2, while the contribution of the electromagnetic fields produced by ρ_1 are contained in the constant K. The values of K are determined from Maxwell's equations and the results are presented in Appendix A.

The transverse position of each particle in the beam can be found by solving the equations of motion with some known initial conditions. Knowing the position of every particle in the beam, in principle, we can construct the charge density of the beam. A self-consistent solution is obtained if the charge density we constructed is the same as the one we assumed (ρ in eq. 2.3). A convenient method by which to obtain this self-consistent solution is to find the particle distribution function in phase space.

III. Self-consistent Distribution Functions in Phase Space

In this section we proceed to find the self-consistent particle distribution functions in phase space, $\psi(x, y, z, P_x, P_y, P_z, t)$, which give rise to the charge densities assumed in Section II. In analogy to Eq. (2.3) we write

$$\psi = \left[\psi_0(x, y, P_x, P_y) + \xi\psi(x, y, P_x, P_y)\right]\lambda\,\delta(P_z - P_{z0}) , \tag{3.1}$$

where ψ_0 and ψ_1 are the particle distribution functions corresponding to the charge densities ρ_0 and ρ_1, i.e.,

$$\rho_0 = e\int \psi_0 \lambda\, \delta(P_z - P_{z0})\, d^3p \tag{3.2}$$

and

$$\rho_1 = e\int \psi_1 \lambda\, \delta(P_z - P_{z0})\, d^3p \tag{3.3}$$

In general, the particle distribution function satisfies the Vlasov equation

$$\frac{\partial\psi}{\partial t} + \dot{x}\frac{\partial\psi}{\partial x} + \dot{y}\frac{\partial\psi}{\partial y} + \dot{z}\frac{\partial\psi}{\partial z} + \dot{P}_x\frac{\partial\psi}{\partial P_x} + \dot{P}_y\frac{\partial\psi}{\partial P_y} + \dot{P}_z\frac{\partial\psi}{\partial P_z} = 0 . \tag{3.4}$$

When we substitute eqs. (2.5) for the time derivatives of the coordinates and momenta and eq. (3.1) for ψ into the Vlasov equation, we obtain to first order in ξ

$$P_x\frac{\partial\psi_0}{\partial x} + P_y\frac{\partial\psi_0}{\partial y} - \omega_0^2 x\frac{\partial\psi_0}{\partial P_x} - \omega_0^2 y\frac{\partial\psi_0}{\partial P_y} = 0, \tag{3.5}$$

and

$$i(kv - \omega)\psi_1 + P_x\frac{\partial\psi_1}{\partial x} + P_y\frac{\partial\psi_1}{\partial y} - \omega_0^2 x\frac{\partial\psi_1}{\partial P_x} - \omega_0^2 y\frac{\partial\psi_1}{\partial y} = -K\left(x\frac{\partial\psi_0}{\partial P_x} \pm y\frac{\partial\psi_0}{\partial P_y}\right) , \tag{3.6}$$

with the upper and low signs for the monopole and quadrupole oscillations, and k = 0 for the bunched beam.

When a self-consistent solution ψ_0 that satisfies eqs. (3.2) and (3.5) is inserted into eq. (3.6), a self-consistent solution ψ_1 that satisfies eqs. (3.3) and (3.6) is found with the oscillation frequency, ω, obeying the dispersion relation

$$4\omega_0^2 - (kv - \omega)^2 = Ka \quad . \qquad (3.7)$$

The consequences of this dispersion relation are explored in the next section.

IV. Consequences of the Dispersion Relation

In this section the dispersion relation, eq. (3.7), is analyzed. The values of k are restricted to $kv = n\Omega$, with Ω the revolution frequency, n a positive integer for the uniform beam, and n equal to zero for the bunched beam. It is convenient to write $\omega_0 = \nu_0 \Omega$ with ν_0 the unperturbed number of betatron oscillations per revolution, so that we obtain for the dispersion relation:

$$4\nu_0^2\Omega^2 - (n\Omega - \omega)^2 = a\left[K_r(\omega) + i K_i(\omega)\right] , \qquad (4.1)$$

where K_r and K_i denote the real and imaginary parts of K. In practice $a\sqrt{K_r^2 + K_i^2} << (\nu_0\Omega)^2$ so that two of the roots of eq. (4.1) for ω are:

$$\omega = (n \pm 2\nu_0)\Omega \mp \frac{a}{4\nu_0\Omega}\left[K_r(n\Omega \pm 2\nu_0\Omega) + i K_i(n\Omega \pm 2\nu_0\Omega)\right] \qquad (4.2)$$

For the case of the uniform beam the sign of the imaginary part of $K(\omega)$ is determined by the sign of ω, so that

$$\omega = (n \pm 2\nu_0)\Omega \mp \frac{a K_r(|n\Omega \pm 2\nu_0\Omega|)}{4\nu_0\Omega} \mp i \,\mathrm{sign}(n \pm 2\nu_0) \frac{a K_i(|n\Omega \pm 2\nu_0\Omega|)}{4\nu_0\Omega} , \qquad (4.3)$$

with the upper sign representing fast wave, the lower sign representing a slow wave, and $K_i(|n\Omega \pm 2\nu_0\Omega|) > 0$. Since the motion is damped when imaginary part of ω is negative, the fast wave is always damped, while the slow wave is damped only for $n < 2\nu_0$. However for $n > 2\nu_0$ the slow wave grows exponentially with an e-folding time τ given by

$$\tau = \frac{4\nu_0\Omega}{a K_i(|n\Omega - 2\nu_0\Omega|)} \qquad (4.4)$$

For the bunched beam $K_r(\omega) = K_r(-\omega)$ and $K_i(\omega) = -K_i(-\omega)$ so that the dispersion relation is given by

$$\omega = \pm\left(2\nu_0\Omega - \frac{a K_r(2\nu_0\Omega)}{4\nu_0\Omega}\right) - i \frac{a K_i(2\nu_0\Omega)}{4\nu_0\Omega} \qquad (4.5)$$

Hence oscillation is damped if $K_i(2\nu_0\Omega) > 0$ and grows exponentially for $K_i(2\nu_0\Omega) < 0$ with an e folding time, τ, given by

$$\tau = \frac{-4\nu_0\Omega}{a K_i(2\nu_0\Omega)} \qquad (4.6)$$

Unstable monopole oscillations have very long growth times (many years) and thus impose no practical limitations on the design of accelerators and storage rings. However, for unstable quadrupole oscillations the growth times are short enough to be of practical importance. Hence, we restrict our attention to only quadrupole oscillations. We find for a single bunched beam that quadrupole oscillations are always stable if

$$m < 2\nu_0 < \left(m + \tfrac{1}{2}\right) \qquad (4.7)$$

with m an integer. For unstable quadrupole oscillations the maximum growth rate, $1/\tau_2$, is related to the maximum growth rate of the unstable dipole oscillations of Laslett, Neil, and Sessler,[1] $1/\tau_1$, by:

$$\frac{1}{\tau_2} = \left(\frac{a}{b}\right)^2 \frac{1}{\tau_1} \quad . \qquad (4.8)$$

In general, the actual growth rate is less that $(1/\tau)$ calculated from eq. (4.6) because, in the analysis thus far, all of the particles have been assumed to have the same unperturbed frequency, ω_0, i.e., Landau damping has not been considered. The effects of Landay damping will be discussed in the next section.

V. Landau Damping of Quadrupole Oscillations

If there is a sufficiently large spread in the unperturbed oscillation frequency, ω_0, it is possible to have stability as a consequence of Landau damping for cases in which the stable condition eq. (4.7) is not satisfied. A spread in ω_0 may be achieved by a spread in the betatron oscillation frequency, ν_0, due to a variation in the oscillation amplitudes of the particles, or a spread in the revolution frequency, Ω, due to an energy spread. Under these conditions, the dispersion relation for quadrupole oscillations of a uniform beam is

$$1 = \int_0^\infty\!\!\int_0^\infty \frac{4\pi a^2 \omega_0 \, \beta_0'(a) f(E)\left[u + (1+i)V\right] da \, dE}{e\lambda\left[4\omega_0^2 - (n\Omega - \omega)^2\right]} \qquad (5.1)$$

with

$$u = -\frac{\lambda r_0 c^2 (b^2 - a^2)}{\omega_0 \gamma^3 a^2 b^2}\left(\frac{b^2 + a^2}{2b^2}\right) \qquad (5.2)$$

and

$$V = \frac{\lambda r_0 \beta^2 c^3}{\sqrt{2\pi\omega\sigma}\; b^3 \omega_0}\left(\frac{a^2}{b^2}\right), \qquad (5.3)$$

where $f(E)$ and $\rho_0(r)$ are the energy and radial distribution functions of the particles, normalized such that

$$\int_0^\infty f(E)\,dE = 1 \quad \text{and} \quad 2\pi\int_0^\infty r\rho_0(r)\,dr = \lambda$$

and the other parameters are defined in Appendix A. This dispersion relation is similar to the dispersion relation that has been treated exhaustively by Laslett, Neil, and Sessler.[1] For the case where the spread in the quantity, S, defined by

$$S = (n - 2\nu_0)\,\Omega$$

is larger than $\sqrt{u^2 + v^2}$, the unstable oscillations are Landau damped. But, if the spread in S is smaller than $\sqrt{u^2 + v^2}$ and $n > 2\nu_0$ the oscillations are unstable and the maximum growth rates are given by

$$\frac{1}{\tau_m} = \frac{\lambda r_0 \beta^2 c^3 a^2}{\nu_0 \Omega b^5 \sqrt{2\pi\sigma(n-2\nu_0)\Omega}}\;. \qquad (5.4)$$

A similar dispersion relation holds for the quadrupole oscillations of a tightly bunched beam ($\nu L/2\pi R \ll 1\pi$). For this assumption the oscillations are Landau damped when the spread in $2\nu_0\Omega$ is larger than $\sqrt{A_r^2 + A_i^2}$, where

$$A_r = -\frac{N r_0 c^2 (b^2 - a^2)}{\pi^{\frac{1}{2}} \nu_0 \Omega L \gamma^3 b^2 a^2}\left(\frac{b^2 + a^2}{2b^2}\right) + \frac{N r_0 \beta^2 c^3 a^2}{2\pi \nu_0 \Omega \gamma b^5 \sqrt{\pi\beta\sigma c L}}\left[\Gamma\!\left(\tfrac{1}{4}\right) + \sqrt{\frac{2L}{R}}\sum_{n=1}^{\infty}\frac{\cos 4\pi n\nu_0}{n^{\frac{1}{2}}}\right] \qquad (5.5)$$

and

$$A_i = \frac{N r_0 \beta^2 c^3 a^2}{2\pi \nu_0 \Omega \gamma b^5 \sqrt{\pi\beta\sigma c R}}\left[\nu\sqrt{\frac{L}{R}} + \sqrt{2}\sum_{n=1}^{\infty}\frac{\sin 4\pi n\nu_0}{n^{\frac{1}{2}}}\right],$$

But if the spread in $2\nu_0\Omega$ is less than $\sqrt{A_r^2 + A_i^2}$ and $(m-1/2) < 2\nu_0 < m$, the quadrupole oscillations are unstable and the maximum growth rates are given by

$$\frac{1}{\tau_m} = -A_i\;. \qquad (5.6)$$

VI. Numerical Examples and Observations

In most accelerators and storage rings, such as the Brookhaven AGS, Argonne ZGS and SLAC Ring (proposed), the cross sections of the beam and vacuum chamber are not circular. Therefore, the theory must be extended before it can be rigorously applied. However, for the purpose of illustration, we calculate the spread in $(2\nu_0\Omega)$ necessary to Landau damp quadrupole oscillations in these machines and the maximum growth rate, using the machine and beam parameters given below.

	AGS	ZGS	SLAC RING	
R =	10^4	3×10^3	3×10^3	cm
L =	1.3×10^3	5.9×10^2	3×10	cm
ν_0 =	8.5	0.8	5.25	
γ =	1.5	3	6×10^3	
λ =	1.3×10^8	3.2×10^8	3.7×10^9	cm^{-1}
σ =	1016	10^{16}	0.4×10^{18}	sec^{-1}
a =	3	7.5	1	cm
b =	6	12.5	5	cm

For these parameters, the local damping fields restrict the range of unstable oscillations in the AGS to $8.49 < \nu_0 < 8.5$ or $8.99 \le \nu_0 \le 9$. While a spread in $2\nu_0\Omega$ equal to $\sqrt{A_r^2 + A_i^2}$ will produce Landau damping with

$$A_i = -.816 \times 10^4 + .93\,\mathrm{Re}G(2\nu_0),$$

$$A_i = 7.04 + .93\,\mathrm{Re}G(2\nu_0),$$

and

$$G(x) = \sum_{n=1}^{\infty}\frac{\sqrt{2}\,e^{i2\pi nx}}{\sqrt{n}}$$

A plot of $\mathrm{Re}G(x)$ and $\mathrm{Im}G(x)$ is shown in Fig. 1. The maximum growth rate is

$$\frac{1}{\tau_m} = -A_i\ \sec^{-1}\;.$$

Similar calculations are made for the ZGS and SLAC Ring. For ZGS, oscillations may be unstable when $0.86 < \nu_0 < 1$ or $0.36 < \nu_0 < 0.5$, with

$$A_r = -0.95 \times 10^3 + 0.55\,\mathrm{Re}G(2\nu_0)$$

and

$$A_i = 0.46 + 0.55\,\mathrm{Im}G(2\nu_0).$$

For SLAC Ring, unstable oscillations may occur when $5.41 < \nu_0 < 5.5$ or $5.91 < \nu_0 < 6.0$, with

$$A_r = 0.471 + 0.013\,\mathrm{Re}G(2\nu_0)$$

and

$$A_i = 0.0168 + 0.013\,\mathrm{Im}G(2\,\nu_o).$$

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Appendix A

The perturbed electromagnetic fields and hence the perturbed forces on a particle are calculated from Maxwell's equation. The value of K determined by the force is

$$K = \frac{F_x}{m_o \gamma \xi x} \tag{A.1}$$

For the uniform beam we obtain for the monopole and quadrupole oscillations:

$$K = \frac{-4 r_o \lambda c^2}{\gamma^3 a^3} + (1 + i\,\mathrm{Sign}\,\omega)\frac{r_o \lambda a}{b}\left|\frac{\omega}{2\pi\sigma}\right|^{\frac{1}{2}}\left[\left|\frac{\omega}{c}\right|\left(K - \frac{\omega\beta}{c}\right)^2\right], \quad \text{(monopole)} \tag{A.2}$$

$$K = \frac{-2 r_o \lambda c^2 (b^4 - a^4)}{\gamma^3 b^4 a^3} + (1 + i\,\mathrm{Sign}\,\omega)\frac{4 r_o \lambda \beta^2 c^3}{\gamma b^5}\left|\frac{1}{2\pi\sigma\omega}\right|^{\frac{1}{2}}, \quad \text{(quadrupole)} \tag{A.3}$$

and for the bunched beam we obtain:

$$K = \frac{-4 r_o N c^2}{\sqrt{\pi}\,\gamma^3 a^3 L} + \frac{r_o N a \beta^2 c^3}{\pi b \gamma \sqrt{\pi\beta c \sigma L^9}}\left\{\left[\frac{\Gamma(\frac{9}{4})}{\gamma^4} - \frac{\nu^2 L^2}{R^2}\frac{\Gamma(\frac{9}{4})}{\gamma^2} + \frac{\nu^2 L^2\,\Gamma(\frac{5}{4})}{R^2\,4}\right] + \frac{i\nu L}{R}\left[\frac{\Gamma(\frac{11}{4})}{\gamma^4} - \frac{\Gamma(\frac{7}{4})}{\gamma^2} + \frac{\nu^2 L^2}{R^2}\frac{\Gamma(\frac{7}{4})}{4}\right]\right. \quad \text{(monopole)} \tag{A.4}$$

$$+ \frac{3 r_o N a \beta^2 c^3 2\pi^{\frac{1}{2}}}{64\pi b \gamma \sqrt{\pi\beta c \sigma R^9}} \sum_{n=1}^{\infty}\left(\frac{35}{4\gamma^4 (2\pi n)^{9/2}} - \frac{i\,5\nu}{\gamma^2 (2\pi n)^{7/2}} - \frac{\nu^2}{(2\pi n)^{5/2}}\right) e^{i 2\pi n \nu}$$

$$K = \frac{-2 r_o N (b^4 - a^4) c^2}{\pi^{1/2} L \gamma^3 b^4 a^3} + \frac{2 N r_o \beta^2 c^3 a}{\gamma \pi b^5 \sqrt{\pi\beta c \sigma L}}\left[\Gamma(\tfrac{1}{4}) + \frac{i\nu L}{R}\,\Gamma(\tfrac{3}{4})\right]$$

$$+ \frac{2 r_o N \beta^2 c^3 a}{\pi \gamma b^5 \sqrt{\pi\beta c \sigma R}}\, 2\pi^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{e^{i 2\pi n \nu}}{(2\pi n)^{1/2}} \quad \text{(quadrupole)} \tag{A.5}$$

where
r_o = classical radius of the particles
L = length of the bunch
R = radius of the machine
ν = ω/Ω
σ = conductivity of the pipe

and Γ (x) is the complete gamma function.

References

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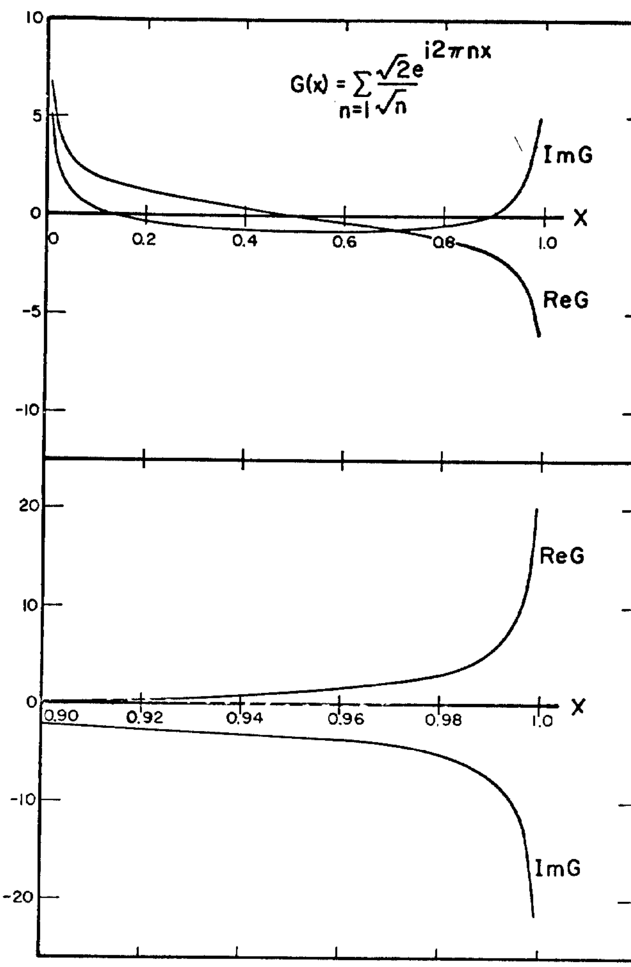

G(x) versus X

Fig. 1.