SPACE-CHARGE EFFECTS ON THE QUADRUPOLE FOCUSING SYSTEM IN LOW-ENERGY PROTON LINEAR ACCELERATORS*

Shoroku Ohnuma and Joseph N. Vitale
Physics Department, Yale University
New Haven, Connecticut

Summary. A general computer program has been used to calculate the stability diagram for (+)(-)(+)(-) quadrupole arrangements when the space charge is uniformly distributed in an elliptic cylinder or an ellipsoid. Linearized equations for longitudinal and transverse envelopes of the bunch have been solved simultaneously in order to find the velocity dependence of the phase damping.

I. Introduction

In the past few years, a number of studies have been made for the effects of space charge in a beam transport system and in a low-energy proton linear accelerator.[1-12] Most of these investigations are, however, primarily on the longitudinal phase stability in proton linacs and the limiting current due to the reduction of the phase stable area.

For space-charge effects on the transverse motion of the particle, Kapchinskij and Vladimirskij[1] assumed a uniformly distributed elliptic cylinder and calculated betatron oscillation parameters[13] for quadrupole focusing systems in proton linacs. Catura[4] and Crandall[10] also used a cylinder for their numerical studies of transverse motions assuming uniform as well as non-uniform charge distributions. Their results are, however, confined to drift spaces with essentially no focusing elements involved. A uniformly distributed ellipsoidal bunch has been assumed by Lapostolle[5] in his formulation of space charge effects on longitudinal and transverse motions in a linac and a method for applying this formalism in a numerical computation has been discussed in detail.

It is quite obvious that the motion of a particle bunch in a proton linac under the influence of a strong space-charge force is quite complicated and that its entire picture could not be found without performing a detailed numerical calculation. This would certainly involve an orbit tracing of hundreds of particles or equivalent rings and disks.[5,10,11] Even for the study of the longitudinal phase motion alone, it is not at all clear whether one can simply represent the transverse motion by one or two constant parameters.[3,7,9] For a given focusing system, the transverse envelope of the particle bunch is affected by its longitudinal size so that, at least in principle, one cannot control the transverse motion without first knowing the phase motion throughout the linac. The work reported here was motivated by the belief that (1) the design of a focusing system for high-intensity proton linacs must take the space-charge effect into account and (2) an approximate phase motion coupled to the transverse motion through the space charge is necessary for this design. It is also hoped that this approximate phase motion could be used as a guidance of a detailed study on the problem of phase damping.[14]

The present work is never intended to replace detailed numerical calculations. There are a number of inevitable limitations and some of them may turn out to be serious. These limitations are: (1) Particles are assumed to be uniformly distributed in either an elliptic cylinder for a continuous beam or in an ellipsoid for a bunched beam. (2) No effects of image charges due to the conducting wall are included. (3) All particles are assumed to get the same defocusing action from the accelerating field. This means that the important coupling effects previously studied in detail[15-18] are not included. (4) Longitudinal phase motion is linearized and the ellipsoidal shape of the bunch is assumed to be maintained throughout the cavity. (5) Fields due to neighboring bunches are not considered. (6) Difference of particle velocities in a bunch is neglected.

II. Potential Due to Space Charge

For a continuous beam along the z-direction whose cross section is an ellipse in the x-y plane, the electric field due to a uniformly distributed space charge is[1]

$$E_{x(y)} = \frac{1}{4\pi\varepsilon_o}\,\frac{4I}{v}\,\frac{x(y)}{r_{x(y)}(r_x+r_y)} \tag{1}$$

where $2r_x$ and $2r_y$ are the size of the beam cross section, I is the current, v is the (common) velocity of particles, and

$$(1/4\pi\varepsilon_o) = 8.988 \times 10^9 \text{ (m·ohm/sec)}.$$

If a bunch is assumed to be a three-dimensional ellipsoid, again uniformly distributed, the electric field can be expressed approximately in the form[5]

$$E_{x(y)} = \frac{1}{4\pi\varepsilon_o}\cdot\frac{6I}{v}\cdot X_B\cdot(1-f)\,\frac{1}{r_{x(y)}(r_x+r_y)}\,x(y) \tag{2}$$

$$E_z = \frac{1}{4\pi\varepsilon_o}\,\frac{6I}{v}\,X_B\,f\,\frac{1}{r_x r_y}\cdot z\,. \tag{3}$$

Here, X_B is the bunching factor defined by

$$X_B = v\lambda/2cr_z = 2\pi/2(\Delta\varphi) \tag{4}$$

where $2r_z$ and $2(\Delta\varphi)$ are the longitudinal size of the bunch and the corresponding total phase spread, respectively, when the wave length of the accelerating field is λ. The parameter f is a function of $r_z/\sqrt{r_x r_y}$ and its explicit form is

$$f(p) = \frac{1}{1-p^2}\left(1 - \frac{p}{\sqrt{1-p^2}}\tan^{-1}\frac{\sqrt{1-p^2}}{p}\right) .$$

For $p > 1$, this is equivalent to

$$f(p) = \frac{q^2}{1-q^2}\left(\frac{1}{\sqrt{1-q^2}}\ln\frac{1+\sqrt{1-q^2}}{q} - 1\right)$$

with $q \equiv 1/p < 1$. Also $f(p=1) = 1/3$.

There are a number of objections to this particular choice of the space charge field. Particle distributions in the four- or six-dimensional phase space are quite unrealistic.[1,19,20] The assumption that the bunch is symmetric about $z = 0$ (or the synchronous point) is not valid in a linac unless the phase spread is very small. Also, because of magnet misalignments, the longitudinal axis of the bunch can be different from the axis of the linac and the effect of the image charge may become serious.[12] On the other hand, fields taken here are linear in coordinates and simplify the analysis considerably. For three-dimensional cases, any choice other than the uniformly distributed ellipsoid makes the equation of motion for (x,y,z) nonlinear and nonseparable.[7] It must also be noted that these particle distributions are self-consistent.[7,20]

III. Equations for Beam Envelopes

Equations for beam envelopes (or beam profiles) were first introduced by Courant and Snyder[13] in their classical work on the theory of the alternating-gradient synchrotron. They are best suited to study the motion of a particle bunch as a whole rather than the orbit of individual particles when the equation of motion is linear. This is certainly a very good approximation for the transverse motion in linacs and could also be used for the longitudinal phase oscillation with a small amplitude.

It is convenient to take the dimensionless parameter $\underline{n}$ as the independent variable,

$$(dn/ds) = (c/v_s\lambda) = (1/\beta_s\lambda) \qquad (5)$$

where $\underline{s}$ is the distance along the longitudinal axis and $v_s = c\beta_s$ is the velocity common to all particles. In a linac, it is natural to take λ equal to the wavelength of the accelerating field, making $\underline{n}$ equivalent to the cell number. Let $u \equiv x$, y, or z of individual particles. Because of the linear approximation, the equation of motion takes the form

$$u'' = - K_u \cdot u \ . \qquad (6)$$

The general solution of Eq. (6) is

$$u(n) = Aw(n)\cos[\psi(n) + \psi_0] \qquad (7)$$

where A and ψ_0 are arbitrary constants fixed by the initial conditions and

$$w'' = - K_u\cdot w + (1/w)^3, \quad \psi' = (1/w)^2 \ . \qquad (8)$$

If the beam consists of all particles with the same value of A but different values of the initial phase ψ_0, it is represented by an ellipse

$$(u/w)^2 + (wu' - w'u)^2 = A^2 \equiv F \qquad (9)$$

in (u,u') phase space. The largest value of u at each point $\underline{n}$, which gives the size of the beam in each direction, is then $u_m = Aw(n)$ and the equation for u_m is

$$u_m'' = - K_u u_m + A^4/u_m^3 \qquad (10)$$

For a strictly periodic structure, it is always possible to find a periodic solution of Eq. (8). For $u = x$ or y, the periodic solution $w(n)$ is related to the betatron oscillation parameter[13] $\beta(s)$

$$w(n) = \sqrt{\beta(s)/\beta_s\lambda} \ . \qquad (11)$$

The ellipse (9) then represents a "matched" beam and the largest transverse beam size is Aw_{max}. When the beam occupies the area πW in $(u, du/ds)$ space,

$$A^2 \equiv F = \lambda(\beta_s W) \ . \qquad (12)$$

If the system is not periodic, the initial beam shape can still be represented by the ellipse (9) with initial values $w(0)$ and $w'(0)$. The change of the beam size $u_m(n)$ are then determined by solving Eqs. (8) or (10).

For transverse directions, $u_m = r_x$ or r_y and

$$r''_{x(y)} = - Q_{x(y)} r_{x(y)} + F^2_{x(y)}/r^3_{x(y)} + K_c \qquad (13)$$

The parameters F_x and F_y are related to beam qualities, Eq. (12); in most cases, they have the same value F. The parameters Q_x and Q_y represent the action of quadrupole magnets. If the field gradient of the quadrupole magnet is H' (kG/cm),

$$Q_{x(y)} = \pm \frac{H'}{0.3130}\beta_s\lambda^2 \equiv \pm g^2 \qquad (14)$$

where λ is in meters and plus (minus) sign is for the focusing (defocusing) direction. The defocusing (or focusing) action of the accelerating field in linacs can be approximated by a single impulse at the center of each cell which introduces a discontinuity in $r'_{x(y)}$

$$r'_{x(y)} \rightarrow r'_{x(y)} + \Omega r_{x(y)}$$

with

$$\Omega = - (\pi e E_0 T\lambda)/(m_0 c^2 \beta_s)\sin\varphi \qquad (15)$$

where E_0 is the average field on the axis, T is the transit time factor, and φ is the phase of each particle. Although the value of Ω should be different for different particles in the beam giving rise to an important coupling effect,[15-18] it is here assumed to be the same for all particles in the bunch. Effects of space charge are represented by the function K_c,

$$K_c = (e\lambda^2)/(m_0c^2)\,\frac{1}{4\pi\varepsilon_0}\,\frac{4J}{v}\,\frac{1}{r_x + r_y} \qquad (16)$$

where

$J = I$ for continuous beams
$= 3X_B I\,(1-f)/2$ for bunched beams.

For the longitudinal direction, $u_m = r_z$ and

$$r_z'' = -K_z r_z + F_z^2/r_z^3 + K_{cz} \quad (17)$$

$$K_z = -(2\pi e E_o T\lambda)/(m_o c^2 \beta_s)\sin\varphi_s \quad (18)$$

$$K_{cz} = (e\lambda^2)/(m_o c^2)\left(\frac{1}{4\pi\varepsilon_o}\frac{6I}{v}X_B f\cdot r_z\right)/(r_x r_y) \quad (19)$$

The initial shape of the bunch in (z,z') phase space can be fixed by $r_z(0)$, $r_z'(0)$, and F_z,

$$F_z(z/r_z)^2 + (r_z z' - r_z' z)^2/F_z = F_z \ . \quad (20)$$

IV. Stability Diagram and Betatron Oscillation Parameters

When the focusing system is strictly periodic, its characteristics are completely determined by the stability diagram. For a slowly varying system, the investigation of the "static" stability diagram is usually an essential first step of the design.[21] For this purpose, it is more convenient to solve differential equations for dimensionless quantities w_x and w_y instead of the beam size r_x and r_y. From Eq. (13),

$$w''_{x(y)} = -Q_{x(y)} w_{x(y)} + 1/w^3_{x(y)} + q/(w_x + w_y) \quad (21)$$

where the dimensionless parameter q representing the space charge effect is given by

$$q = (e\lambda^2)/(m_o c^2)(1/4\pi\varepsilon_o)\frac{4J}{v}\frac{1}{F} \ . \quad (22)$$

For a bunched beam, J is a function of r_z and $\sqrt{r_x r_y}$ (through the dependence on X_B and f) as well as of the current I. However, the change of r_z and $\sqrt{r_x r_y}$ within one period of the system is very small so that q can be regarded as a constant parameter in the calculation of the "static" stability diagram. Betatron oscillation parameters $\beta(s)$, $\alpha(s)$ and μ (phase advance of the betatron oscillation per focusing period) of Courant-Snyder are then obtained from the periodic solution w(n), w'(n) of Eq. (21) through the relation (11) and

$$\alpha(s) = -w(n)\,w'(n) \quad (23)$$

$$\mu = \int ds/\beta(s) = \int dn/w^2(n) \text{ for one period.} \quad (24)$$

The quadrupole system (+)(-)(+)(-) has been studied extensively for a wide range of Ω and q values assuming the magnet length and the distance between magnets equal to $(\beta_s\lambda/2)$ and $(\beta_s\lambda)$, respectively. Part of these results are shown in Figs. 1 and 2.

Several interesting features of the space charge effect should be mentioned here. Some of them have already been observed by Kapchinskij and Vladimirskij.[1] (1) For a given set of (Ω, g), the dependence of the parameter w_{max} on the space charge parameter q can be approximately given by

$$w_{max} = A_o + A_1 q \text{ for small values of q}$$
$$= B_o + B_1\sqrt{q} \text{ for large values of q}$$

where A_o, A_1, B_o, and B_1 are all positive and $B_o \ll B_1$. It should be noted here that q is proportional to the current and inversely proportional to the transverse phase space area of the beam. [See Eq. (22).] (2) The parameter ψ which was introduced by Smith and Gluckstern,[21]

$$\psi = w_{max}/w_{min} \quad (25)$$

is almost independent of Ω and q. An empirical formula

$$\psi = 1.556 - 1.032g + 0.691g^2$$
$$1.0 \lesssim g \lesssim 1.7 \ , \quad 0 \lesssim \Omega \lesssim 0.1 \ , \quad (26)$$

is sufficiently accurate for most applications. Throughout the system, it is a good approximation to take

$$r_x r_y = (\beta_s \lambda W)(w^2_{max}/\psi) \ . \quad (27)$$

(3) The phase advance of betatron oscillations per focusing period, μ, depends strongly on the parameter q. The dependence of μ on (g,q) is shown in Fig. 3 for $\Omega = 0$. The wave number $k_t = \mu/2\beta_s\lambda$ will be quite different from the value for q = 0 that was used previously in evaluating coupling effects.[15,17] (4) When the length of each quadrupole magnet is different from $(\beta_s\lambda/2)$, the increase or decrease in g necessary to get the same value of w_{max} (for a given set of Ω and q) can be estimated from the relation

$$(g_\varepsilon/g_{\varepsilon=1/2}) = \left[2/(3-2\varepsilon)\right]^{1/4}/\sqrt{2\varepsilon} \quad (28)$$

where g_ε corresponds to the magnet length $\varepsilon\beta_s\lambda$.

As an example of possible applications, the dependence of max r_x (or r_y) at 0.75 MeV on the value of the quadrupole field gradient H' are given in Figs. 4 and 5. Parameters chosen here correspond approximately to design values of the high-intensity (100 ~ 200 mA) injector for AGS at Brookhaven and of the Los Alamos linac for its meson physics facilities.

V. Longitudinal Beam Envelopes - Linear Approximation

In the calculation of the betatron oscillation parameters, the longitudinal phase spread of the bunch (or, equivalently, X_B) is assumed to be constant. However, this assumption cannot be justified in low-energy proton linacs. Space charge effects would make the particle bunch expand in the longitudinal as well as transverse directions. At the same time, there will be a compensating effect due to the accelerating field which is responsible for the familiar (velocity)$^{-3/4}$ damping of the phase oscillation. In order to find a complete picture of the accelerated bunch, it is then necessary to solve Eqs. (13) and (17) simultaneously. The primary purpose of this study is to get a semi-quantitative estimate of space charge effects on the phase damping when the total phase spread of the bunch is relatively small as in the Los Alamos linac.

Parameters chosen for the model linac cavity are listed in Table 1. The invariant transverse phase space area $\beta_s W$ is always 0.1π cm-mrad. The initial shapes of the bunch in the longitudinal phase space are

$$0.688 \times 10^{-4} (\Delta\varphi)^2 - 2 \times 0.0633 (\Delta\varphi)(\Delta\gamma) + 1.460 \times 10^4 (\Delta\gamma)^2 = 0.603 \times 10^{-5}$$

for I = 21.6 mA and

$$0.535 \times 10^{-4} (\Delta\varphi)^2 - 2 \times 0.175 (\Delta\varphi)(\Delta\gamma) + 1.926 \times 10^4 (\Delta\gamma)^2 = 0.633 \times 10^{-5}$$

for I = 100 mA. Variations of the phase spread (half size) and $\sqrt{r_x r_y}$ along the linac are shown in Figs. 6 and 7. Two curves, (A) and (B) in Fig. 6, and (C) and (D) in Fig. 7, correspond to two different variations of the quadrupole strength. When the current is small, there is a very small amount of "pulsation" due to the space charge in the phase spread and one can easily find the focusing system which produces a desirable variation of $\sqrt{r_x r_y}$. The final phase spread is about 10-12% larger than the case with no current. With the choice $\sqrt{r_x r_y} \gtrsim 0.4$ cm, the space charge effect does not seem to change the phase damping substantially. The situation is entirely different for I = 100 mA. The longitudinal motion is unstable at the beginning and the bunch starts expanding rapidly. Unless quadrupoles are properly adjusted to take care of the decrease in space charge effects, the focusing system becomes too strong and $\sqrt{r_x r_y}$ goes down. Thereafter, variations of the phase spread and $\sqrt{r_x r_y}$ are coupled in a complicated manner and one sees, as in the curve (C) of Fig. 7, a large "pulsation" in both longitudinal and transverse directions. It is possible to choose the strength of quadrupoles so that the transverse size $\sqrt{r_x r_y}$ changes smoothly along the linac. This is shown as curve (D) in Fig. 7 although the last "pulsation" beyond cell No. 38 is still not completely eliminated. It should be noticed here that variations of the transverse size strongly affect the variation of the total phase spread. The final phase spread could easily be twice of what one expects with no current. Also, the final phase spread depends not only on the final value of $\sqrt{r_x r_y}$ but on the variation of $\sqrt{r_x r_y}$ along the linac. For example, if the linac used here were terminated at cell No. 39, the final phase spread would be 22° or 30° depending on the choice of the quadrupole strength (C) or (D). However, both choices give the same final value (0.51 cm) for $\sqrt{r_x r_y}$. In principle, one could substantially reduce the "pulsation" of the phase spread (except the first one) by introducing a special variation in $\sqrt{r_x r_y}$ which, however, would be quite unrealistic. The point to be emphasized here is that the flexibility in the quadrupole focusing system is useful not only for the obvious control of the transverse motion but also for modifying the phase spread of the beam.

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Table 1

	(I)	(II)
Initial Energy	0.75 MeV	0.75 MeV
Final Energy	9.47 MeV	9.85 MeV
Energy Gain per Length	.88-1.48 MeV/m	.90-1.52 MeV/m
Synchronous Phase	-28.8°	-25.8°
Peak Current	21.6 mA	100 mA
Transverse Size		
Major	0.48-0.50 cm	0.55-0.71 cm
Minor	0.36-0.38 cm	0.41-0.52 cm

Total Phase Spread

Initial	17.0°x2	20.0°x2
Final	6.5°(5.7°)*x2	9.8°(6.6°)*x2
Maximum	17.0° at 0.75 MeV	26.5° at 1.16 MeV

* Corresponding values when the current is zero.

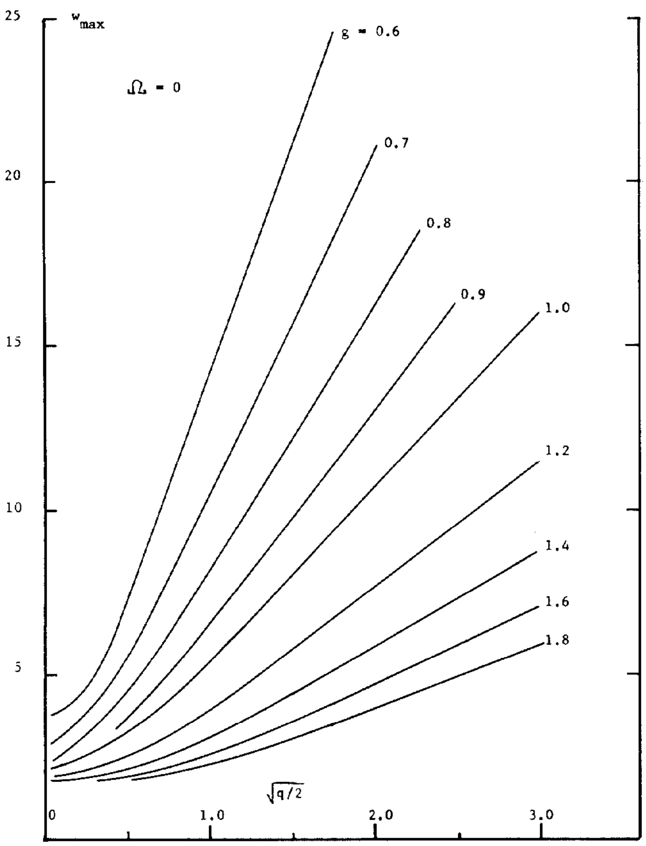

Fig. 1. Maximum betatron oscillation parameters w_{max} of (+)(-)(+)(-) quadrupole systems as a function of the space-charge parameter $\sqrt{q/2}$ for $\Omega = 0$ and g = 0.6 - 1.8. For definitions of g, Ω, and q, see Eqs. (14), (15), and (22), respectively. The maximum transverse size r_x (or r_y) of the beam is $w_{max}\sqrt{\lambda(\beta_s W)}$ where $\beta_s W$ is the invariant phase space area.

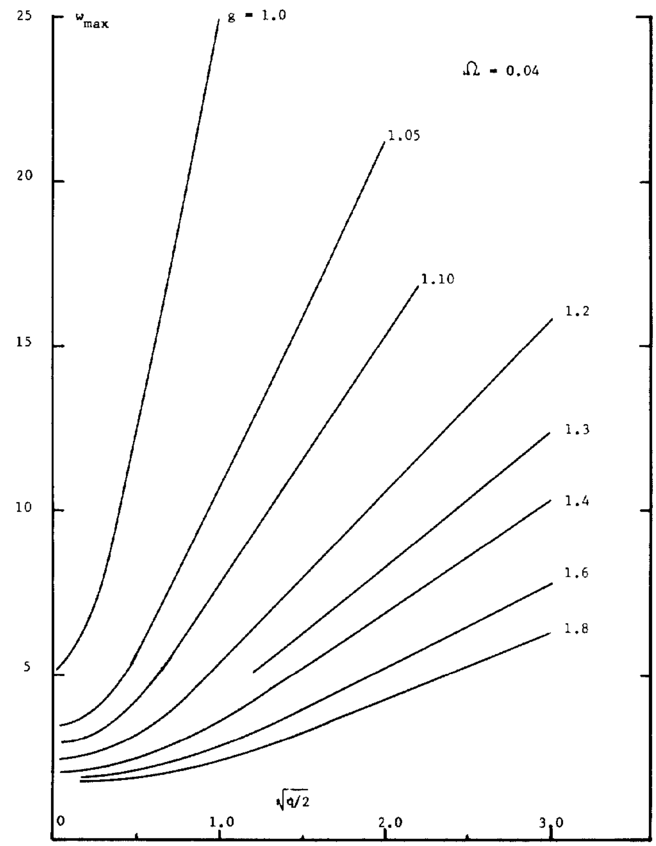

Fig. 2. Same as Fig. 1 with $\Omega = 0.04$ and g = 1.0 - 1.8.

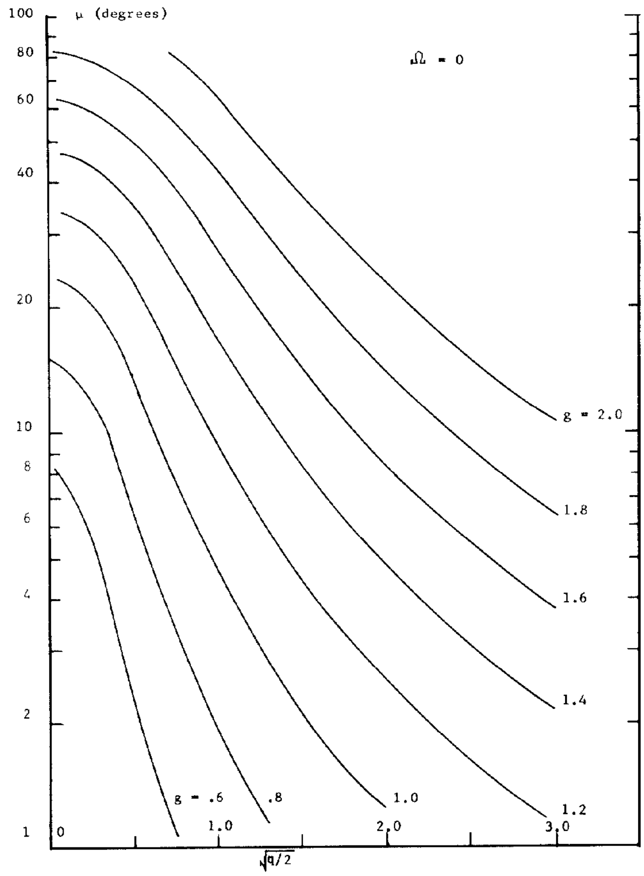

Fig. 3. The phase advance μ (degrees) of the betatron oscillation per focusing period $2\beta_s\lambda$ for $\Omega = 0$ and g = 0.6 - 2.0. The wave number k_t of the transverse motion is $k_t = (\mu/2\beta_s\lambda)$.

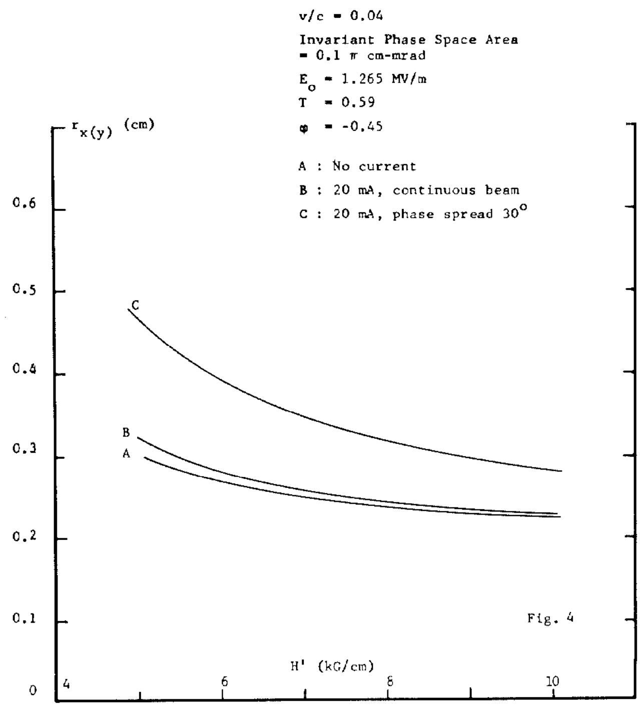

Fig. 4. Maximum transverse dimensions r_x (or r_y) near the injection energy (0.75 MeV) as a function of H', the quadrupole field gradient. Parameters are similar to those for the proposed Los Alamos linac.

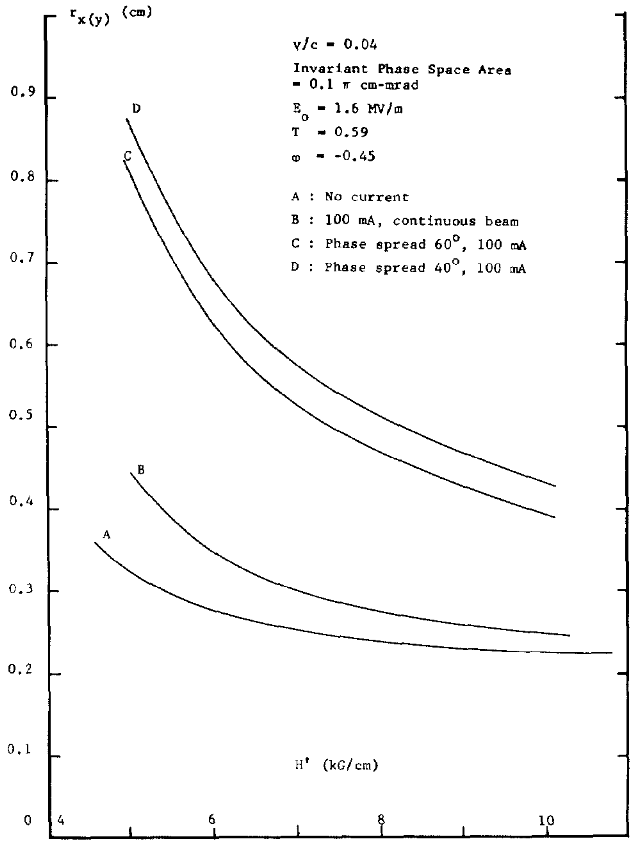

Fig. 5. Same as Fig. 4 for a higher current (100 mA) which corresponds to the new AGS injector linac at Brookhaven.

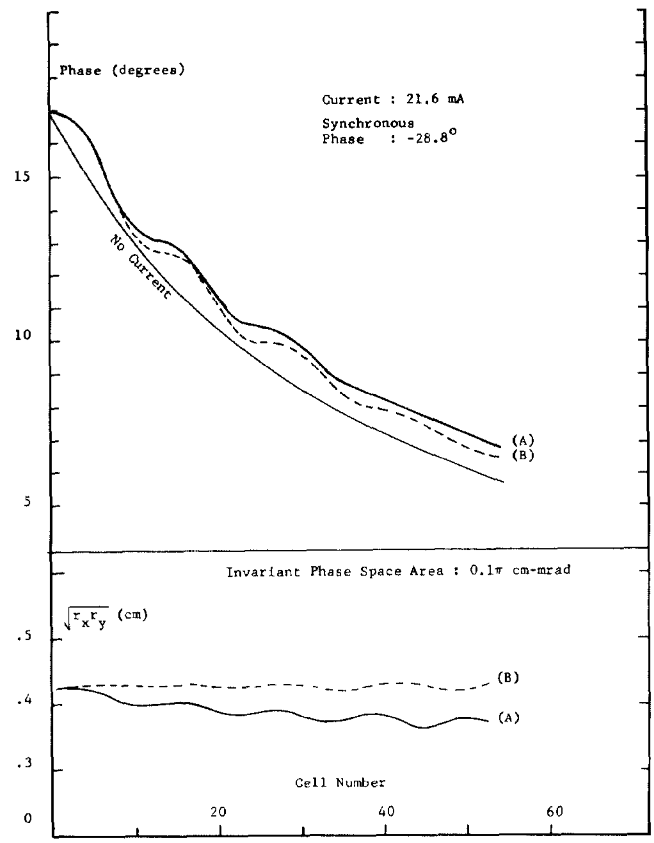

Fig. 6. Variations of the phase spread (half size) and $\sqrt{r_x r_y}$ along the model linac cavity. For machine parameters, see Case (I) of Table 1. Curves (A) and (B) correspond to two different quadrupole strength distributions.

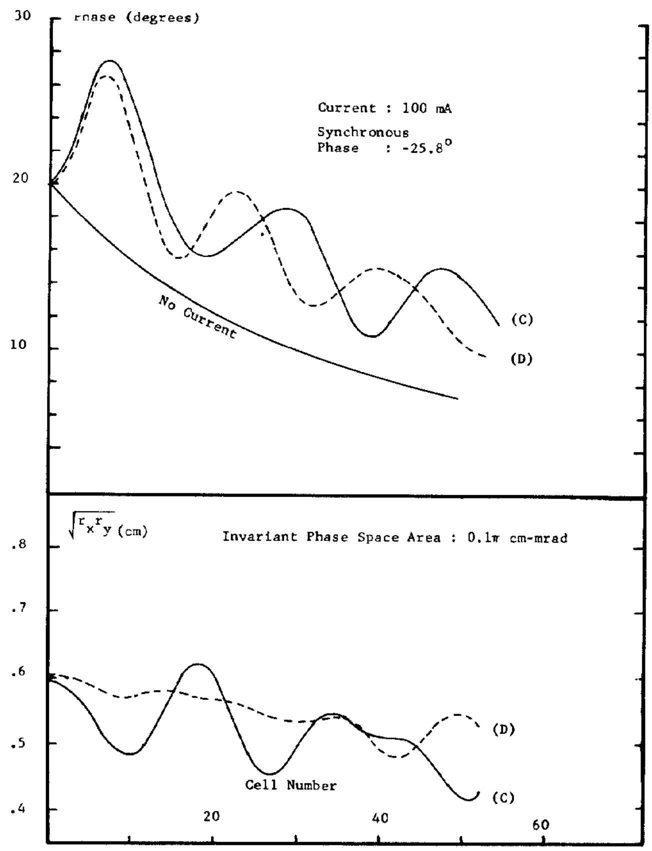

Fig. 7. Same as Fig. 6 for a higher current (100 mA). See Case (II) of Table 1.