BEAM-ENVELOPE OSCILLATIONS WITH SPACE CHARGE IN CIRCULAR PARTICLE ACCELERATORS

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Summary

Properties of matched beams (definition: beams whose dimensions oscillate with the periodicity of the machine) and their stability, are discussed. At intensities beyond resonances several matched solutions exist. It is shown that it may be possible to achieve an increase in the number of accelerated particles by injecting at those intensities and by crossing the resonances subsequently during acceleration.

Introduction

As in previous studies of the problem (1,2,3,4), the assumption is made that any 2-dimensional projection of the 4-dim. transverse density distribution is constant inside an ellipse and zero outside. With this distribution space-charge forces correspond to a beam-dimension dependent constant-gradient field, defocusing in both transverse directions. The particle motion in the y-direction is given by

$$\frac{d^2y}{d\theta^2} + \left[Q_y^{\,2} + \sum_{M=1}^{\infty} A_{yM} \cos (M\theta - \varphi_{yM}) - \frac{e^2NR}{2\pi^2 v^2 \varepsilon_o m \gamma^2 B} \, \frac{1}{r_y(r_z + r_y)} \right] y = 0 \qquad (1)$$

N : number of protons in the machine,
v : particle velocity, A_{yM}: amplitude of Mth, harmonic gradient perturbation,
φ_{yM}: phase of this harmonic, θ azimuthal angle, m : mass of particles, B: longitudinal bunching factor Q_y: zero-intensity Q-value in y-plane, r_y and r_z: semi-axies of the elliptic beam cross-section.

A similar equation is valid for the z-motion.

As the forces are linear in displacement from the centre it is possible to write two different equations for the semi-axies r_y and r_z. Normalizing with respect to the emittances, one obtains:

$$\frac{d^2Y}{d\,\theta^2} + \left[Q_y^{\,2} + \sum_{M=1}^{\infty} A_{yM} \cos (M\theta - \varphi_{yM}) \right] Y - \frac{1}{Y^3} - \frac{2\,\delta}{\sqrt{\frac{E_y}{E_z}}\, Y + Z} = 0 \qquad (2)$$

$$\frac{d^2Z}{d\theta^2} + \left[Q_z^2 + \sum_{M=1}^{\infty} A_{zM} \cos (M\theta - \varphi_{zM}) \right] Z - \frac{1}{Z^3} - \frac{2\,\delta}{Y + \sqrt{\frac{E_z}{E_y}}\, Z} = 0 \qquad (3)$$

$$Y = \frac{r_y}{\sqrt{RE_y/\pi}} \quad (4) \qquad Z = \frac{r_z}{\sqrt{RE_z/\pi}} \quad (5)$$

$$\delta = \frac{e^2\,N}{4\pi\,\varepsilon_o m_o c^2 \beta^2 \gamma^3 B \sqrt{E_y E_z}} \qquad (6)$$

$E_{y,z}$: emittances in y-and z-planes.

Constant-Energy Beams

Unperturbed Beams ($A_{yM} = A_{zM} = 0$)

Although gradient perturbations are essential for stopbands, it is instructive as far as the non-linearity is concerned (dependance of the envelope oscillation frequencies on the oscillation amplitudes) first to consider unperturbed cases.

Figure 1 gives, in the case of matched beams, curves for maximum values for Y and Z, normalized ($Y_{MAX\ REL}$ and $Z_{MAX\ REL}$) with respect to their matched constant values. These constant values are

$$Y = Z = \sqrt{\frac{\delta}{2Q^2} + \sqrt{\left(\frac{\delta}{2Q^2}\right)^2 + \frac{1}{Q^2}}}$$

when $E_z/E_y = 1$ and $Q_z = Q_y = Q$. (In more general cases they cannot be expressed algebraically but increase in a similar way with δ).

Suppose $E_z/E_y = 1$ and $Q_z = Q_y = 6.4$. For $\delta = 0$ the number of envelope oscillations, Q_E, around the matched constant value equals $2 \times 6.4 = 12.8$ per turn. When δ increases Q_E decreases (increased space-charge defocussing) At certain values of δ, Q_E becomes integral, i.e. the envelope, oscillating with the periodicity of the machine, is now matched. With an increasing oscillation amplitude and δ = const., the average space-charge defocusing diminishes. Therefore, to keep Q_E integral (matched beam) when envelope oscillations increase, δ must increase too; curves a and b are bent to the right. Along a the 2 semi-radii oscillate in antiphase (antisymmetrical solution), along b in phase (at the same frequency). As δ continues to increase, Q_E crosses lower integral valuesWhere more matched solutions appear.

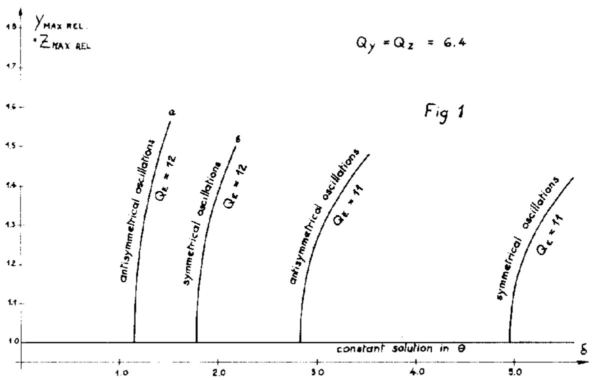


Fig 1

The second figure exhibits in a different way the same non-linearity. The evolution of mismatches are plotted at the end of each of many successive perturbation periods in an envelope phase plane. Each mismatch follows a closed line. The constant matched solution corresponds to point A, the oscillating matched solutions to curve C.

(Outside C : $Q_E > 12$, inside C : $Q_E < 12$, infinitely far away $Q_E = 2 \times 6.4 = 12.8$).

A very general case, where Y- and Z-movements are necessarily different, is presented in Fig. 7 where the dotted lines show the maximum values of the oscillating unperturbed matched beam in units of the non-oscillating beam. The plane with the small emittance and the low Q-value participates strongly at the antisymmetric resonance. The resonance appears as forced in the other plane. At the symmetric resonance the roles are inversed.

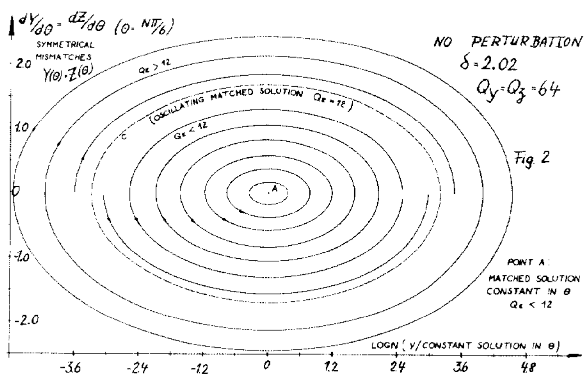


Fig 2

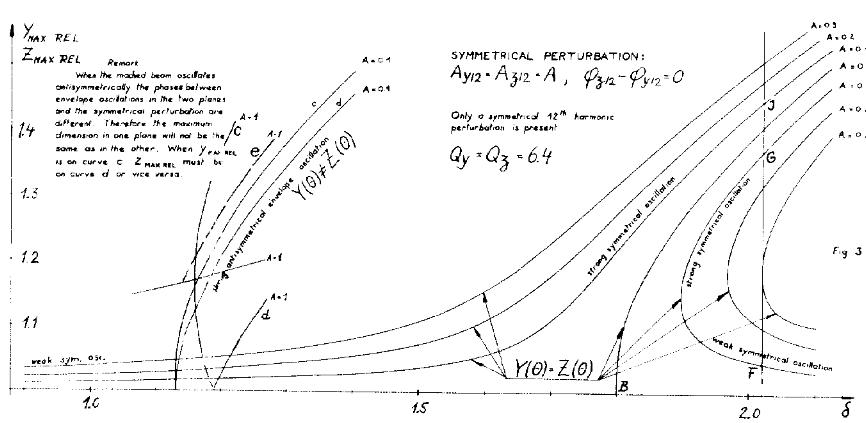


Fig 3

Gradient-perturbed Beams ($Q_y=Q_z$, $E_z/E_y=1$)

Introducing, around the envelope resonance 12, a symmetric gradient perturbation with the most perturbing harmonic (i.e. $A_{y12}=A_{z12}$,[a] $\varphi_{y12}=\varphi_{z12}=\varphi_{12}$) one notices the following (see Fig. 3).

There is at least one matched solution $Y(\theta) = Z(\theta)$ for any δ. The previously non-oscillating envelope solution is slightly modulated by the perturbation. The symmetrically oscillating branches are each divided into 2. On the higher branch the phase between oscillation and perturbation is such that Q_E is increased; as a compensation the envelope now oscillates more. On the lower branch it is the opposite.

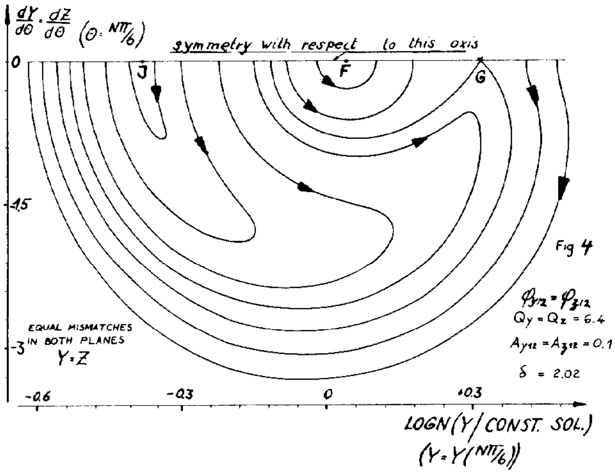
Fig. 4 shows similarly, as Fig. 2, the evolution of equal mismatches in y and z, but for a symmetrically perturbed beam. Points F, G and I represent the matched solutions and correspond to F, G and I, in Fig. 3. I and G are the strongly oscillating solutions. The perturbation phase is such that I is stable and G unstable. F is also stable.

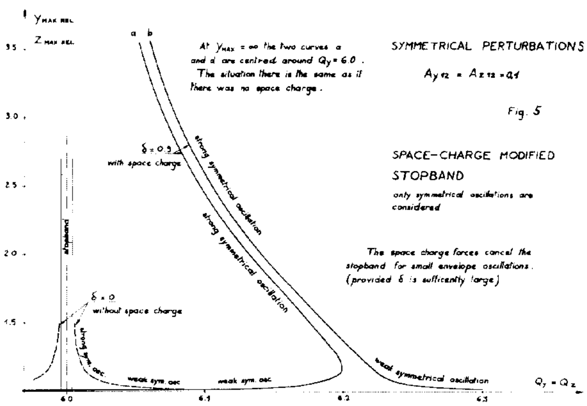
An interesting aspect of the phenomena is offered by Fig. 5, giving $Y_{MAX\ REL}$ for δ constant and a symmetric perturbation as a function of $Q_y = Q_z$. Where at zero-intensity there was a stopband, now 1 stable periodic solution exists.

When oscillating antisymmetrically, the beam presents different maximum dimensions in y and z, due to the perturbation being seen differently. Furthermore, a third matched solution exists[b]. For small oscillations of this solution the phases in the two planes are practically equal; as oscillations increase, however, the phases become asymptotically antisymmetric: for infinitely large oscillations maxima are not at $\theta = (2\pi N + \varphi_{12})/12$ in one plane and $(2\pi(N+1) + \varphi_{12})/12$ in the other, as for the already mentioned antisymmetric solution (branches c and d) but at

a) Zero-intensity stopband widths are $\frac{A_{y,\,z12}}{12}$

b) Solution found by F. Sacherer, Berkeley, Private communication

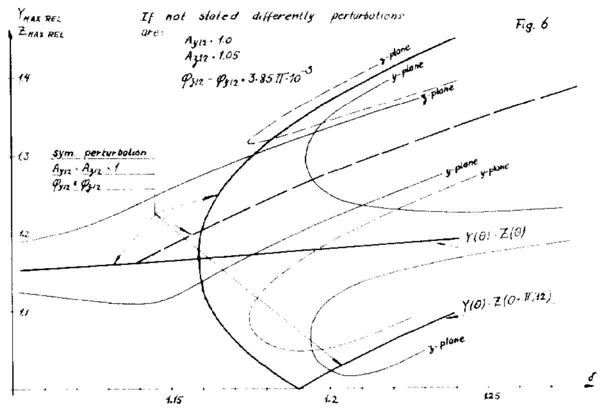




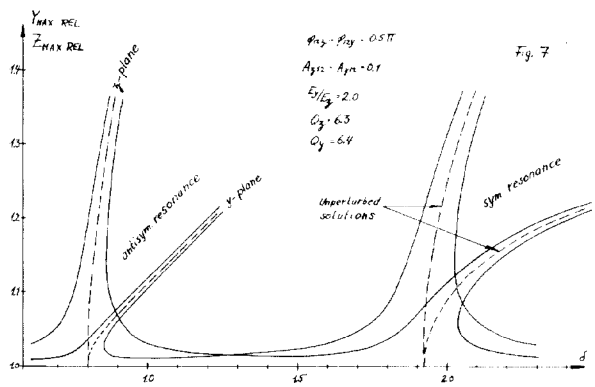
$(2\pi\ (N + \frac{1}{2}) + \varphi_{12})\ /12$ and $(2\pi\ (N - \frac{1}{2}) + \varphi_{12})\ /12$. Branch e (dotted lines), Fig. 3, shows this solution for a large perturbation (A = 1). When the perturbation vanishes, branches c, d and e of Fig. 3 coincide and become branch a in Fig. 1.

In the case of a purely antisymmetric (12th harmonic) perturbation ($A_{y12} = A_{z12}$, $\varphi_{y12} = \varphi_{z12} + \pi/12$), there is always a matched antisymmetric solution $Y(\theta) = Z(\theta + \pi/12)$. Curves like c, d and e of Fig. 3 would be at the symmetric resonance ($\delta \approx 1.8$). At the antisymmetric resonance ($\delta \approx 1.15$) there would be continuity between the weakly oscillating antisymmetric solution and the strongly oscillating antisymmetric solution in analogy with the continuity at $\delta \approx 1.8$ in Fig. 3.

If, in addition to a large symmetric perturbation, other perturbations with arbitrary phases but very small amplitudes are introduced, the curves for the matched beams double in number and become rather complicated, as can be seen from Fig. 6. At a given δ above 1.15 there are now 5 different matched solutions.



Fortunately for the treatment of more general cases, the solutions drawn with dotted lines in Fig. 6 appear to move to the upper right when $(\varphi_{y12} - \varphi_{z12})/\pi$ assumes non-integral values or $|A_{z12}|$ becomes different from $|A_{y12}|$. This way, they are of little relevance as long as beam dimensions do not grow excessively. Fig. 7 shows curves for low-dimension matched beams in a very general case.



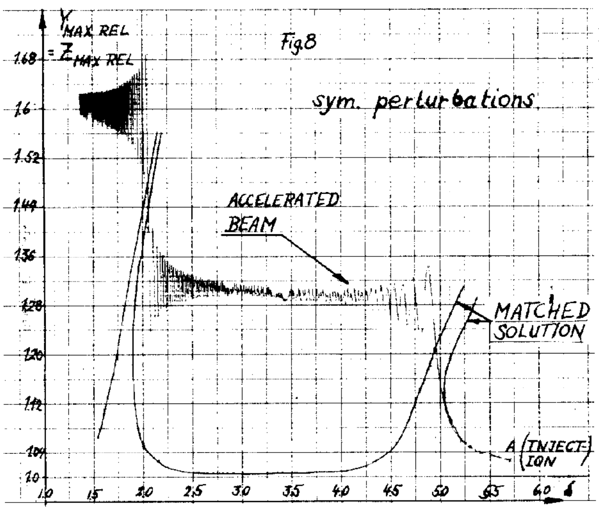
Resonance Crossings (variable energy)

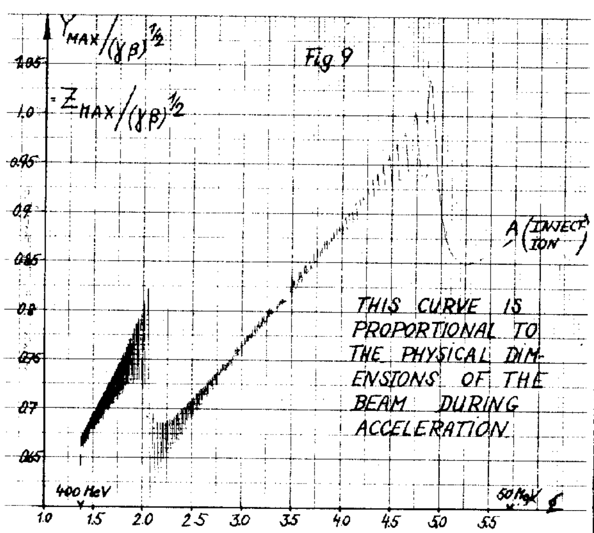
Incoherent betatron resonances can be crossed with circulating beams by changing Q_y or Q_z or, if δ at injection is sufficiently high, during acceleration: although during acceleration transverse beam dimensions shrink as $(\gamma\beta)^{-1/2}$ the importance of space-charge forces diminishes compared with that of the increasing magnetic fields. Supposing $B \sim \gamma^{-3/4}$, it can be shown from (6) that δ varies with $\frac{1}{\beta} \times \left(\frac{1}{\gamma}\right)^{5/4}$. Thus, if a beam is injected at a δ-value above resonances, during acceleration those resonances are likely to be crossed.

Two-Resonance Crossing

Fig. 8 shows in the plane $Y_{MAX\ REL}$ =

$Z_{MAX\ REL}$ the crossing with a beam, initially matched, starting at $\delta = 5.7$; $E_y/E_z = 1$, $Q_y = Q_z = 6.4$, $A_{y12} = A_{z12} = A_{y11} = A_{z11} = 0.1$. First the envelope remains matched. When it enters the 11th resonance, oscillations increase rapidly bringing the beam through it with an oscillation increase approximately twice what it would be if the beam was also matched later. Due to the mismatch the 12th resonance is crossed with a higher δ than in the matched case. Fig. 9 shows how dimensions are diminished by the factor $(\gamma\beta)^{-1/2}$ during acceleration.

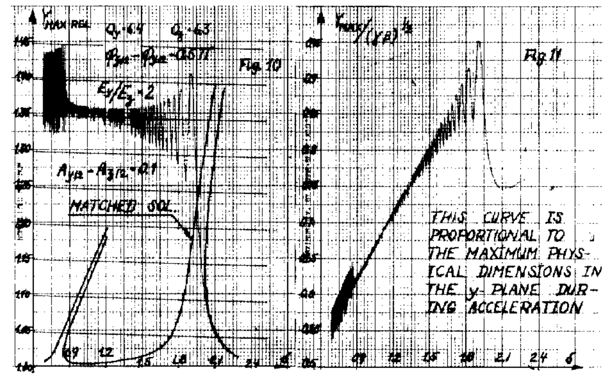


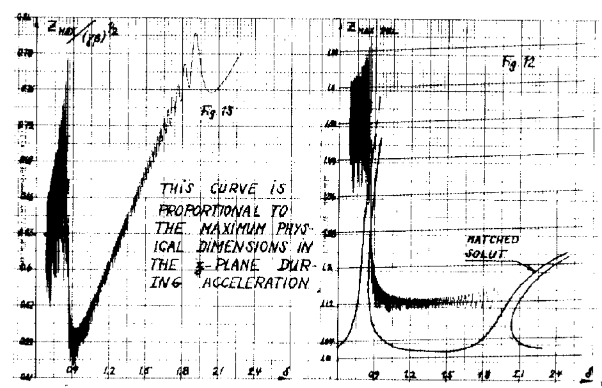


When the same resonances are crossed in the opposite direction (with an increasing δ) the envelope oscillation amplitude tends to ∞, as can be anticipated from the curves for the matched solutions.

General Case

Fig. 10, 11, 12, 13 show crossings of the 12th symmetric and antisymmetric resonances for a beam where $E_y/E_z = 2$, and $Q_y = 6.4$ and $Q_z = 6.3$; the beam is matched at injection. At the symmetric crossing, mainly the y-plane resonates, at the antisymmetric resonance it is mainly the z-plane.





Chamber-Wall Limitations

If N, E_y/E_z, γ and B are kept constant and E_y varies, the constant chamber wall limitations a and b appear as parabolas in the planes Y_{MAX}, δ and Z_{MAX}, δ.

$$Y_{MAX} \quad \frac{a}{\rho_y}\sqrt{\delta} \qquad Z_{MAX} \quad \frac{b}{\rho_z}\sqrt{\delta}$$

$$\rho_{y,z} = \frac{e^2 N \sqrt{\frac{E_{y,z}}{E_{z,y}}}}{4\pi \varepsilon_o m_o c^2 \beta^2 \gamma^3 B}$$

Dividing these equations with the matched unperturbed solutions, which we approximate with the solution for the round beam, we obtain the conditions:

$$Y_{MAX\ REL} \quad \frac{1}{\rho_y \sqrt{\frac{\delta}{2Q_y^{\,2}} + \sqrt{\left(\frac{\delta}{2Q_y}\right)^2 + \frac{1}{Q_y^{\,2}}}}}$$

Similarly for other plane.

δ small: $Y_{MAX\ REL} < \frac{a}{\rho_y} \sqrt{Q_y \delta}$

δ large: $Y_{MAX\ REL} < \frac{a}{\rho_z} Q_y$

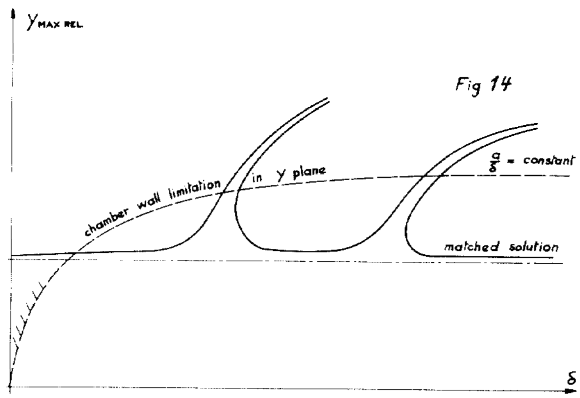


Fig 14

Conclusions

For given intensity, bunching factor, particle energy and chamber dimensions, it is obvious that there is a minimum value for δ : i.e. the emittances cannot be arbitrarily large. If there were no resonances the larger δ would be chosen, the more space there would be between beam and chamber walls. Taking into account resonance blow-ups, it appears interesting to inject matched beams at intensities between two non-linear space-charge induced resonances, only if:

a) the adiabatic beam-dimension shrinkage with $(\gamma\beta)^{-1/2}$ is larger or not very much smaller than the increase at the resonance crossing. This condition imposes a lower limit on δ, once the two resonances have been chosen. (It is not fulfilled in the y-plane for the general case discussed in this paper).

b) δ is not chosen so high that a strongly oscillating matched beam with too large dimensions must be injected. This condition imposes an upper limit for δ.

The way the bunching factor B changes during acceleration (and the resulting variation of δ) might affect the crossing of resonances.

Generally speaking, whether or not there is any sense in injecting at intensities beyond one or more resonances depends mainly on the strength of gradient perturbations.

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