THEORY OF LONGITUDINAL INSTABILITIES IN SYNCHROTRONS, WITH APPLICATIONS†,‡

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Summary

By combining the results of longitudinal instability theory and synchrotron oscillation theory, it is possible to derive two simple and useful expressions for longitudinal instability particle threshold. The first, which holds at injection, is given directly in terms of design parameters; in practical cases it turns out to depend only slightly on injection energy and to be independent of field index and vacuum chamber size, provided only that the vacuum chamber is large enough to contain particle oscillations. The second, which is based on adiabatic bunch behavior, displays the variation of threshold with energy; it decreases monotonically from injection to transition (if a real transition exists) and then increases monotonically. Different regions of this universal curve are relevant, depending upon whether the machine considered is a booster, a weak-focusing machine, or a standard AG synchrotron in which transition is crossed. Practically, it appears that recently proposed boosters will be free from longitudinal instabilities if a little care is exercised in design. It is quite evident that proposed "improved" AG machines will have no problems at injection; the treatment of the transition region is still crude, and further analysis will be required for a definite answer - it appears that there might be trouble at least in Phase II (2×10^{13} protons) unless the capture at injection is extremely efficient.

I. Introduction

When the number of particles in an accelerator beam exceeds a certain value, called the particle threshold, N_{bt}, the accelerator becomes susceptible to longitudinal beam degeneration caused by the amplification of beam irregularities.[1-7] Such a state is called a state of longitudinal instability, and the instability is described as resistive or negative-mass depending upon the mechanism producing it. (Below transition, only the former exists; above transition, the latter predominates.)

The particle threshold, which depends on the energy spread in the beam (and hence on the amplitude of the synchrotron - but not betatron - oscillations) may be written[5]

$$N_{bt} = \frac{(2\Delta r_s)^2}{r_o R} \frac{B\ \pi/2}{1 + (Z_o/30)} \frac{\gamma^2 - 1}{\gamma} \times \left| \gamma_t^2 - \gamma^2 \right| \gamma_t^2 \frac{1}{|U'|\ (1 + V^2/U^2)} \qquad (1)$$

where $2\Delta r_s$ is the full synchrotron width, r_o is the classical radius

$$r_o = \frac{e^2}{m_o c^2} \ (= 1.53 \times 10^{-16} \text{cm for the proton}), \qquad (2)$$

R is the average accelerator radius, B is the bunching factor

$$B \equiv \frac{\text{actual number of particles in the beam}}{\text{number of particles in an equivalent uniform beam}}, \qquad (3)$$

Z_o is the characteristic impedance in ohms between beam and tank considered as a transmission line [= 60 ln (b/a) for a coaxial cylindrical geometry in which b is the inner tank radius and a is the beam radius], γ and γ_t are the energy and transition energy respectively in units of $m_o c^2$. The constant U' is a quantity[4] dependent on the density distribution across the tank and on whether the behavior is above or below transition. The ratio V/U is given by[5] (for a coaxial geometry)

$$\frac{V}{U} = \frac{\sqrt{R}}{b\sqrt{n}\,[1 + 2\ \ln (b/a)]} \sqrt{\frac{\mu c}{2\pi\sigma}} \frac{\beta^{3/2}}{1 - \beta^2}, \qquad (4)$$

where n is the harmonic order of the instability, i.e., the approximate ratio of plasma frequency to particle circulation frequency; for relevant values of n this ratio is very small:

$$(V/U) \ll 1 \qquad (5)$$

If we further define, for convenience,

$$\eta \equiv (1/\gamma_t^2) - (1/\gamma^2) \qquad (6)$$

then we may rewrite (2) in the more compact form

$$N_{bt} = \frac{(2\Delta r_s)^2}{r_o R} \frac{B\ \pi/2}{1 + (Z_o/30)} \frac{1}{|U'|} \gamma(\gamma^2 - 1)\ \gamma_t^4\ |\eta|\ . \qquad (7)$$

We can find the threshold as a function of design parameters if we combine Eq. (7) with the results of synchrotron oscillation theory.[8] It

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‡A more detailed treatment is given in BNL Accelerator Dept. Internal Report AADD-128. Please refer to this for details of derivations and discussions omitted due to space limitations.

can be shown that the maximum radial spread, or half radial synchrotron width at injection is

$$\Delta r_s = \frac{R}{\gamma_t^2}\sqrt{\frac{N e V_m}{\pi E h \eta \beta^2}\left[(\pi - 2\varphi_s)\sin\varphi_s - 2\cos\varphi_s\right]} \quad \text{(at injection)} \qquad (8)$$

where h is the harmonic number, N the number of gaps, V_m the peak gap voltage and φ_s the synchronous phase. We must remember that (8) represents the radial spread only if there are actual particles available for capture at the limiting values of φ, i.e., at injection; during the acceleration process the longitudinal size of the beam shrinks, there are simply no particles out as far as $\varphi = \pi - \varphi_s$, and (8) does not give a true picture of the actual spread available to suppress longitudinal instabilities.

Adiabatically the radial spread Δr varies as

$$\Delta r \cong \frac{D}{\eta^{1/4}\beta\gamma^{3/4}} \exp\left(\pm j \int \Omega\, dt\right) \quad (\eta \neq 0) \qquad (9)$$

while the phase difference varies as

$$\varphi - \varphi_s \cong C\left(\frac{\eta}{\gamma}\right)^{1/4} \exp\left(\pm j \int \Omega\, dt\right) \quad (\eta \neq 0) \qquad (10)$$

where

$$\Omega \equiv \sqrt{-\frac{N e V_m h \omega_o^2 \eta \cos\varphi_s}{2\pi E}} \qquad (11)$$

with

$$\omega_o = c/R \quad . \qquad (12)$$

When η is near zero it is necessary to use one-third order Bessel functions.

We must consider rise rates as well as thresholds, because even if a system is operating above particle threshold, there will be no problem if the rise rate is slow enough. The standard rise rate for negative-mass instabilities, which is an upper limit[5] to the actual rise rate, is given by

$$\frac{1}{\tau_N} = n\, c\left\{\frac{N_b}{2\pi B}\frac{r_o}{R^3}\left[1 + (Z_o/30)\right]\frac{\eta}{\gamma^3}\right\}^{1/2} \qquad (13)$$

where N_b is the actual number of particles in the beam. The standard rise rate for resistive instabilities is given by[5]

$$1/\tau_R = (1/2)\ (V/U)\ (1/\tau_N) \qquad (14)$$

which, from (5), is somewhat less than the negative-mass rise rate.

How many time constants τ_R or τ_N can a machine tolerate? It appears from some experiments carried out at Princeton[6] that the beam fluctuations involved may be primarily those of ordinary shot noise, and that the beam degenerates if the shot noise is amplified so much that the fluctuations in beam current are of the order of magnitude of the current itself. (Similar fluctuations had been observed earlier at the Cosmotron.[7]) This process takes something of the order of five or ten time constants. Hence, since the τ's are minimum times,[5] there will probably be trouble in an accelerator if the duration of the state of instability is more than say fifteen or twenty times the value of the relevant minimum τ. [Note that the amplification of the beam degeneration would be cumulative if successive machines (e.g., a booster and a main synchrotron) happened to amplify any of the same frequency range; however this possibility is unlikely.]

There remains the question of n. One can see from Eq. (13) that the rise rate for negative-mass instabilities is proportional to n, and from (13), (14) and (4) that the rise rate for resistive instabilities varies as $\sqrt{n}$. If the beam degeneration goes on for five or ten time constants, only the highest frequency fluctuations will remain as they are amplified so much faster. There is, however, a cutoff, and hence we can use the cutoff value as a good estimate of n. It appears from the experiments at Princeton[6] that cutoff occurs at about

$$n \sim (R/b) \qquad (15)$$

For this value of n, and reasonable values for the conductivity and dimensions, we find the ratio V/U of Eq. (4) quite small, and hence Eq. (7) is justified.

II. Threshold at Injection

We can now calculate the longitudinal instability threshold at injection by simply substituting Eq. (8) for the radial half-width at injection into Eq. (7) for the threshold, to obtain

$$N_i \equiv N_{bt}^{(inj)} = \frac{2B}{|U'|}\frac{1}{1 + (Z_o/30)}\frac{R}{r_o}\frac{N e V_m}{E_o h}\gamma^2 \times \left[(\pi - 2\varphi_s)\sin\varphi_s - 2\cos\varphi_s\right] \quad \text{(injection)} \qquad (16)$$

where E_o is the rest energy $m_o c^2$.

The interesting thing about (16) is how many parameters it does not depend on, or depends on only slightly. In the first place, all the γ_t terms, including η, have cancelled out, so (16) is the same no matter what type of machine. In the second place, the β terms have cancelled out and the only energy dependence left is the γ^2 term, which does not vary very much within the usual range of injection energies, and which certainly varies less in this range than the space-charge threshold. In the third place - and again in contrast with the case of charge instabilities - the threshold is practically independent of beam or transverse tank dimensions, except that (1) there is a small contribution from the ln term, and (2) the tank must be large enough to contain the beam, including synchrotron and betatron oscillations, effects of fluctuations, and any spreading.

III. Variation of Threshold with Energy: The Universal Curve

Now we look at the variation of longitudinal instability threshold as a function of energy through the acceleration cycle.

We start with the threshold equation (7) as before. But now we assume that $(2\Delta r_s)$ varies with energy adiabatically as described by Eq. (9), and - since except for extreme values the phase plots all have about the same shape - that the bunching factor B varies as $(\varphi - \varphi_s)$ as given by Eq. (10). We assume that neither Z_0 nor $|U'|$ vary with energy. [Actually Z_0 does vary, but it depends logarithmically on the over-all beam radius (due to synchrotron width, betatron width and fluctuations, as well as any possible beam spreading); its variation is small and not certain and we shall neglect it. The same comment applies to $|U'|$, which might have a small variation with energy as the beam profile changed.] Then we can write

$$N_{bt} \propto \gamma^{5/4} \, |\eta|^{3/4} \propto \frac{|1 - (\gamma/\gamma_t)^2|^{3/4}}{(\gamma/\gamma_t)^{1/4}} \qquad (17)$$

We note specifically that Eq. (17), like Eq. (16), contains no β terms and hence in general N_{bt} varies rather slowly near injection. Note also that the only energy dependence is upon the ratio (γ/γ_t). Hence the curve, which is plotted in Fig.1, is universal in the sense that it can be scaled to all accelerators. Different machines correspond merely to different portions of the curve. Note that the curve decreases monotonically to zero at transition, and then increases monotonically. [The first derivative $dN_{bt}/d(\gamma/\gamma_t)$ has no real zeroes for finite real $(\gamma/\gamma_t) \neq 1$. But actually Eq. (17) is inadequate at transition - see Section V.]

For a conventional-gradient machine like the Cosmotron,* the factor γ_t^2 is replaced by $(1 - n_f)\,[1 + (s/m)]$, where n_f is the field index and s/m is the ratio of the length of straight sections to the length of magnet sections, i.e., $1 + (s/m) = R/r$; for the standard field index of 0.6 or so this is less than unity. [For example, in the Cosmotron,[9] $\gamma_t \sim 0.7$.] Hence a machine like the Cosmotron operates in the right hand portion of the universal curve of Fig. 1. Note that the injection value of threshold is a lower limit (i.e., things get better), and the threshold increases during the acceleration process to about ten times its injection value.

The booster proposed by Maschke and Smith[10] for use with the Brookhaven AGS is an alternating-gradient synchrotron of moderate field gradient (resulting in this case in $\gamma_t \sim 2.5$) operating from 50 MeV (γ = 1.05) up to either 200 MeV (γ = 1.21) or about 630 MeV (γ = 1.67). Hence on Fig. 1 it operates in a small range in the left hand portion of the curve, from $(\gamma/\gamma_t) = 0.42$ to $(\gamma/\gamma_t) = 0.48$ or 0.67. So during the restricted acceleration cycle of the booster, the threshold does not vary very much, dropping to about 92% of its injection value for 200-MeV output, and to about 66% of its injection value for 630-MeV output. However, the injection value is an upper limit to the threshold, so if it is used as a measure of stability it should be treated conservatively.

The Brookhaven AGS has a $\gamma_t \sim 8.75$ and operates at present between 50 MeV (γ = 1.05) and 33 GeV (γ = 36.2), whence the range of (γ/γ_t) is from 0.12 to 4.1, almost the entire range shown in Fig. 1. The threshold decreases monotonically to about zero at transition and then increases again (see Section V). Here the value of the threshold at injection is no guarantee of stability - the AGS will always be unstable near transition, and the problem is one of making the time of traverse of the unstable region so small that the beam degeneration will not have had enough time to build up to a dangerous value.

IV. Numerical Applications

We first calculate a rough but useful formula for threshold at injection, bearing in mind that this may or may not be a relevant figure for the machine as a whole.

Consider Eq. (16). Let us try to make it more workable. Assume limits on φ_s at one extreme to be 150° or 30° (which corresponds to a normal working value), and at the other extreme to be 175° or 5° (which corresponds to an attempt to get a very high capture efficiency at injection, perhaps by lowering the rise rate of the external magnetic field or distorting the RF wave). Then the bunching factor B would vary[11] from 0.36 up to 0.56 and the absolute value of the quantity in square brackets would vary from 0.68 to 1.74. If we assume reasonable values of between 2 and 3 for $|U'|$ and say 2 for $(Z_0/30)$, we would have

$$\frac{2B}{|U'|}\,\frac{1}{1 + (Z_0/30)}\,\left|\left[(\pi - 2\varphi_s)\,\sin\varphi_s - 2\cos\varphi_s\right]\right| \simeq 0.054 \text{ to } 0.32 \quad . \qquad (18)$$

Then, if we express the accelerator radius R in meters and the peak cavity voltage V_m in kilovolts, we can rewrite (16) as

$$N_{bt} \simeq (0.38 \text{ to } 2.2) \times 10^{11}\,(R_{\text{meters}})\,\frac{N}{h}\,(V_m\ \text{kV})\,\gamma^2 \qquad \text{(injection)} \qquad (19)$$

where the first figure corresponds to an unenhanced capture system with φ_s = 150° or 30° and to $|U'|$ = 3, and the second figure to values of 175° or 5° and 2 respectively.

The radius of the Cosmotron[9] - and for that matter the approximate radius of most 3-GeV proton machines - is about 11 meters. There is one accelerating gap (N = 1), the RF is at the fundamental (h = 1), and the peak voltage V_m is about 2 kV.

*The Cosmotron is not operating, but it is the prototype of several existing machines. Note, however, that these remarks do not apply to the Argonne ZGS, which has $n_f = 0$ and hence $\gamma_t > 1$.

The value of φ_s is about 30° and injection takes place at about 3.6 MeV. Hence the variable factor in Eq. (19) is of the order of 0.38 to 0.57 and

$$N_{bt} \simeq (8 \text{ to } 13) \times 10^{11} \quad \text{(Cosmotron at Injection)} \tag{20}$$

which was just about the operating value. Hence we must consider the rise time. Since the Cosmotron always operates above transition, the instability will be of the negative-mass type. If we use Eq. (13) with the estimates $B \sim 0.36$, $n \sim 40\text{-}45$, $N_b \sim 10^{12}$, we find that at injection the standard rise time is very fast, about 2 μs, so that more than 10^{12} or so particles would be destroyed quickly before the threshold had a chance to rise.

Consider now the proposed booster,[10] which is to have a radius of 10.8 m and is to accelerate from 50 MeV to (if necessary) 1 GeV in 130 ms. We find that the required energy gain per turn ($N V_m \sin \varphi_s$) is about 2.4 keV. Hence, depending on the value of φ_s, $N V_m$ will vary from 5 to 27.5 keV. Since (except possibly at the top extreme) this can be done with one gap, we assume N = h = 1, whence from Eq. (19), at injection,

$$N_{bt} \simeq (2.3 \text{ to } 70) \times 10^{12} \text{ particles} \quad \text{(booster at injection)} \quad . \tag{21}$$

The very wide range arises because $N V_m \sin \varphi_s$, rather than $N V_m$, is stipulated to be fixed. As discussed in the preceding section, the value of N_{bt} drops slightly as the energy increases. Nevertheless, we see that, <u>with a very efficient bunching system</u>, the machine will be stable not only in Phase I of the Brookhaven AGS Improvement Program (about 10^{13} particles) but in Phase II (2×10^{13} particles) as well. But one cannot use a nominal value of φ_s without any thought.

We might note incidentally the rise time; here the instability will always be resistive. From Eq. (13), with $n \sim 120$, $N_b \sim 2 \times 10^{13}$, we find $\tau_N \sim 0.2$ μs ; from Eq. (4), taking $\sigma \sim 10^{16}$ sec^{-1} and hence $\sqrt{\mu c/2\pi\sigma} \sim 10^{-3}$ cm$^{\frac{1}{2}}$, we find $V/U \sim 2 \times 10^{-5}$ whence from Eq. (14) $\tau_R \sim 20$ ms at injection. The value is about a third of this (three times as fast) at a 630-MeV extraction. These times are enough less than the boosting times (about 80 ms to get up to 630 MeV) that it is indeed necessary to take care to be below the particle threshold.

The situation with the <u>Brookhaven AGS</u> is very different. The threshold is very high, for the machine[12] has a radius of 125 meters, and has N=h and V_m = 15 keV, whence

$$N_{bt} \simeq \begin{Bmatrix} (0.79 \text{ to } 4.6) \\ (1.05 \text{ to } 6.1) \\ (1.67 \text{ to } 9.7) \end{Bmatrix} \times 10^{14} \text{ particles} \begin{Bmatrix} (50 \text{ MeV}) \\ (200 \text{ MeV}) \\ (500 \text{ MeV}) \end{Bmatrix} \quad \text{(AGS at injection)} \tag{22}$$

But the problem is not at injection but at transition. This is a complex calculation indeed. What we shall do here is make rough calculations of (A) how long the AGS is below threshold, and (B) the (minimum) rise time constant of the instability. By comparing these two quantities we can get a rough idea of whether or not there will be longitudinal instability problems.

(A) <u>Duration of the Instability</u> - Let $\delta\gamma$ be the half-width (not to be confused with $\Delta\gamma$, the half-spread of γ in the beam) of γ during which the machine will be unstable. If we assume $\gamma_i \sim 1 \ll \gamma_t$, $\delta\gamma \ll \gamma_t$, then from Eq. (17), we can show that the half-width $\delta\gamma$ depends on the number of particles in the beam and the injection threshold as

$$\delta\gamma \simeq \frac{1}{2} \left(\gamma_t \frac{N_b}{N_i} \right)^{4/3} \quad . \tag{23}$$

Now the energy of the protons increases about 100 MeV in 3 ms,[12] so that half of the total time during which the system is unstable is

$$\delta t = 0.25 \, (N_b/N_i)^{4/3} \text{ seconds} \quad . \tag{24}$$

[The numerical factor in Eq. (24) is <u>not</u> precisely one-fourth, merely 0.25 to two decimal places.] We enter figures in the first row of Table I, where we consider three values of N_b : the present AGS (2×10^{12}), the improved AGS in Phase I (10^{13}) and in Phase II (2×10^{13}). In each case the two figures correspond to extremes in Eq. (22).

	Present $N = 2 \times 10^{12}$	Improved Phase I $N = 10^{13}$	Improved Phase II $N = 2 \times 10^{13}$
Half-time in unstable zone δ_t , ms	1.9 - 0.18	11 - 1.0	15 - 1.4
Minimum rise time τ_N , ms	0.51 - 1.5	0.12 - 0.35	0.074 - 0.22
Ratio δ_t/τ_N	4 - 0.1	90 - 3	200 - 6
Remarks	OK	?	??

TABLE I
AGS Behavior Near Transition

(B) <u>Minimum Rise Time</u> - The instability will exist as a resistive one during the first half of the crossing time and as a negative mass one during the second half. If we assume $n \sim 2250$ and again $\sqrt{\mu c/2\pi\sigma} \sim 10^{-3}$ cm$^{\frac{1}{2}}$, we find from Eq. (4) the value $V/U \sim 0.01$. Hence we simply ignore what happens during the resistive part and just find τ_N from Eq. (13). We see from (13) that the rise rate is zero precisely at transition. Its value at an amount $\delta\gamma$ away is obtained by replacing $1/\gamma^3$ by $1/\gamma_t^3$ and η by $2\delta\gamma/\gamma_t^3$; if we replace the terms in square brackets in Eq. (13) by 3, we get a minimum rise time at the threshold point (which corresponds to N_b particles) of

$$(\tau_N)_{min} \simeq 0.19 \left(\frac{10^{12}}{N_b} \frac{N_i}{N_b} B_{inj} \right)^{1/2} \text{ ms} \qquad (25)$$

where we have used (10) and (23). We calculate this for the cases we are considering and put the result in Table I, second row.

The relevant figure is the number of τ_N time constants in half the unstable zone crossing time; this is listed in the third row of Table I. As we pointed out, if the number in the third row is as much as 15 or 20 there may be trouble.

V. Discussion

The same general conclusion would appear to follow for the AGS as for the proposed boosters - with some care, there will probably be no trouble in Phase I ($N_b = 10^{13}$), and with somewhat more care, there should be no trouble in Phase II ($N_b = 2 \times 10^{13}$). But the injection synchronous phase angle must be chosen closer to 0^o or 180^o than one normally would.

The choice of phase angle is much more important than one would think - particularly in connection with the behavior of the AGS at transition - because of the many ways in which if affects the stability. A larger bunching factor raises the threshold by making the beam more uniform. Further, the corresponding larger synchrotron oscillation amplitude means greater energy spread and hence greater protection against instabilities. And the larger bunching factor means a slower rise rate. More experimental work in this field would be very useful.

This entire AGS transition calculation is of course extremely crude and serves only to point out that there may be problems and that the choice of φ_s is crucial; this crudity comes from assuming that (1) the WKB formulas hold at transition; (2) the energy spread is so small that the system is monoenergetic even at transition; (3) the theory of Neil and Sessler[3] holds at transition; (4) the "average" rise time is two or three times the minimum; (5) the beam breaks up in five or ten time constants; (6) the relevant n is given by Eq. (15). [For a more extensive discussion see BNL AADD-128.]

So a more careful analysis of what happens at transition is the next step. It is important that stronger predictions be made about these longitudinal instabilities. For in contrast to transverse instabilities, they cannot be servoed away, and the only static scheme for suppressing them,[13] which involves coating the inside of the vacuum chamber probably with an oxide, raises problems of vacuum maintenance and radiation damage, as well as breaking down theoretically at the high frequency limit, just where the instability is worst.

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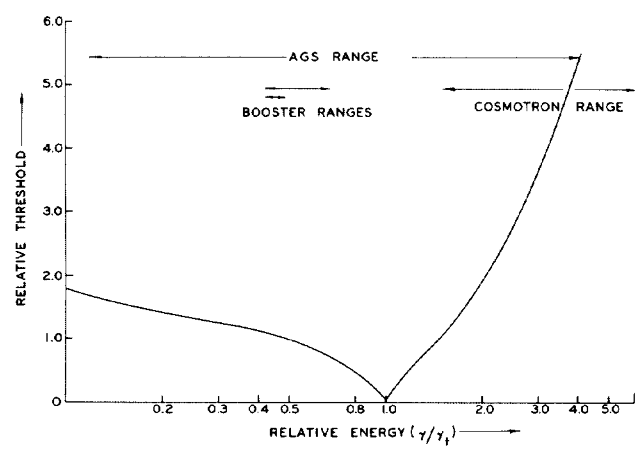

Fig. 1. Universal threshold curve: relative longitudinal instability threshold as a function of energy throughout the acceleration cycle.