REVISED LINAC BEAM DYNAMICS EQUATIONS

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Summary

The set of equations for proton linac beam dynamics given by P. Lapostolle at the Frascati Conference (1965) is derived again by a more rigorous method and extended to include relativistic effects. This is accomplished by a systematic classical perturbation theory using expansions with respect to small perturbation parameters, one describing the influence of the field on the free-particle motion, the other relativistic effects. This permits estimates of the accuracy.

1. Introduction

At the Frascati Conference P. Lapostolle[1,2] gave a set of difference equations, which correct and extend the so-called Panofsky equations[3] describing the phase change and energy gain along a linear accelerator. In this new version transverse motion too was taken into account. But the method used in the derivation involves a slight ambiguity. Here we treat the same problems in a different way which avoids this difficulty. In the course of this we develop a systematic classical perturbation theory based on expansions of the parameter $\bar{E} = eE_1/(m\omega v_o)$ (< 0.1) which characterizes the influence of the field on the free-particle motion. By use of this one is able to separate the equations of motion into parts of different order of magnitude, each order corresponding to a step in solving the equations of motion by iterations starting from free-particle motion. This permits the estimation of the accuracy of the approximation.

We solve the first order equations and give the change in energy, phase and transverse motion across a linac gap. We extend the treatment through a perturbation parameter $E_c = eE_o/(m\omega c)$ to include relativistic effects and to estimate their influence.

2. Homogeneous Time-Harmonic Field

The motion of a proton in such a field served as a simple example to study the method[4]. It is there where the importance and usefulness of the perturbation parameter

$$\bar{E} = eE_1/(m\dot{z}_o\omega^2) = (eE_1/\omega)/(m\, dz_o/dt) < 0.1 \qquad (1)$$

(= impulse exerted by the field/free-particle momentum) was found. For the estimate $E_1 \leq 14$ MV/m, $\omega = 2\pi \times 200$ MHz, $dz/dt \sim 0.5$ MeV protons have been taken. By use of it one may expand energy and phase into powers of $\bar{E}$ and compare with the general results given by Lapostolle[2,4].

One may even get numerically an exact solution for the energy gain ΔW^{ex} and compare with 1st order ΔW_1 (~ transit time factor) and 2nd order approximations. $1 \leq \Delta W_1/\Delta W^{ex} \leq 1.01$, $1 \leq (\Delta W_1 + \Delta W_2)/\Delta W^{ex} \leq 1.002$ for $\bar{E} = 0.1$ and $15^o \leq -\varphi_o \leq 45^o$. By the same method we investigated how the choice of the reference point for the phase and the velocity influences the accuracy of the approximate formulae. These come closest to exact results for reference points near the centre of the gap.

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3. Representation of the General Field

We express the axially symmetric TM-field which is also symmetric with respect to the centre of the gap z = 0, by a Fourier integral:

$$E_z(z,r,\varphi) = (E_1/2\pi)\cos(\varphi + \varphi_o) \int_{-\infty}^{\infty} dk_z\, b(k_z)\, e^{ik_z z} J_o(\gamma r)/J_o(\gamma a) \qquad (2)$$

where

$$\bar{V}(r,\varphi) = V(r)\cos(\varphi + \varphi_o) = \int_{-\infty}^{\infty} E_z(z,r,\varphi)\,dz =$$

$$= E_1 \cos(\varphi + \varphi)_o\, b(0) J_o(k_o r)/J_o(k_o a) \qquad (3)$$

is the voltage along the line r = const., $V(0) = V_o$, $V(a) = V_1 = E_1 g$; a = radius of the drift tube bore and:

$$k_o = \omega/c \quad \gamma(k_z) = (k_o^2 - k_z^2)^{1/2} = ik_r(k_z) \qquad (4)$$

For E_r, B_Θ one has to replace $J_o(\gamma r)$ in (6) by $-ik_z J_1(\gamma r)/\gamma$, $-k_o J_1(\gamma r)/(\gamma c) \times tg(\varphi + \varphi_o)$ respectively.

We require $Im\gamma > 0$. Describing gap plus drift tubes by a circular wave guide (radius a) where along a circumferential slot of the wall an (reasonable) electrical field:

$$E_z^o(z) = E_1 \sum_{s=\infty}^{-\infty} B_s \cos(2\pi s z/g) \quad |z| \leq g/2 \qquad (5)$$

($B_o = 1$, $B_{-s} = B_s$) is applied then:

$$b(k_z) = 2\sum_{s=\infty}^{-\infty} B_s(-1)^s \sin(k_z g/2)(k_z + 2\pi s/g)^{-1},$$

$$B_s \sim s^{-2} \qquad (6)$$

In most cases we shall not use (5) and (6) explicitly, they serve mainly to obtain an idea about the analytic properties of the amplitude $b(k_z)$. The integrals are all single-valued.

Transit Time Factor: We define it as:

$$\bar{V}(0,\varphi)\ T(k_z, r) = \int_{-\infty}^{\infty} E_z(z, r, \varphi) \cos(k_z z)\, dz \quad (7)$$

By the use of E_z in (2) it turns out:

$$T(k_z,r) = T_o(k_z)\ I_o(k_r r) \quad (8)$$

$$T_o(k_z) = b(k_z) J_o(k_o a)/[g I_o(k_r a)] \quad (9)$$

For a homogeneous field along the slot ($B_s = 0$, $s \neq 0$) it is:

$$T_o(k_z) = [\sin(k_z g/2)/(k_z g/2)] \times [J_o(k_o a)/I_o(k_o a)] \quad (10)$$

but $J_o(k_o a) \approx 1$ in most cases. With (9) we may write (2):

$$E_z = (V_o/2\pi)\cos(\varphi+\varphi_o)\int_{-\infty}^{\infty} T_o(k_z) I_o(k_r r)\cos(k_z z) dk_z \quad (11)$$

For E_r and B_Θ insert $k_z I_1(k_r r)/k_r$, $-k_o\ I_1(k_r r)/(ck_r)\ tg(\varphi+\varphi_o)$ instead of $I_o(k_r r)$. The symbols I_n are the modified Bessel functions. Equation (11) agrees with the usual definition[1,2].

S-coefficient: We define:

$$\bar{V}(0,\varphi)\ S(k_z,r) = 2\int_{0}^{\infty} E_z(z,r,\varphi)\sin(k_z z)\, dz \quad (12)$$

However, $S(k_z,r)$ does not factorize as T in (8), at variance with earlier assumptions[1,2]. Employing (6) in $E_z(z,r,\varphi)$ one gets the formula stated in Table III. Only an approximate expression of S for homogeneous field has been given before[5].

4. Non-relativistic Motion

The Perturbation Equations. It will be shown later that the influence of the magnetic field is of the same magnitude as other relativistic contributions and both are small for low energy. Therefore, only the electrical field will be included here in the equations of motion. They are (dots denote derivations with respect to phase $\varphi = \omega t$):

$$k\ddot{z} = (\bar{E}/2\pi)\cos(\varphi+\varphi_o)\int dk_z b(k_z) J_o(\gamma r) e^{ik_z z}/J_o(\gamma a) \quad (13)$$

and similarly for $k\ddot{r}$ by the substitutions described after (4). The path C coincides with the real k_z-axis, except it is indented upwards (downwards) at $k_z = -k(+k)$ for later use. The data are:

$$\varphi = 0 : z = 0 \quad \dot{z} = \dot{z}_o = k^{-1} \quad r = r_o \quad \dot{r} = \dot{r}_o \quad (14)$$

Inserting the perturbation series:

$$r = r^{(o)} + \bar{E}\, r^{(1)} + \ldots \qquad z = z^{(o)} + \bar{E}\, z^{(1)} + \ldots \quad (15)$$

into (13), expanding in powers of $\bar{E}$ and comparing equal powers of $\bar{E}$ gives a set of equations:

$$\bar{E}^o : \ddot{r}^{(o)} = 0 \qquad \ddot{z}^{(o)} = 0 \quad (16)$$

$$\bar{E} : 2\pi k \ddot{z}^{(1)} = \int_C dk_z b(k_z)\, e^{ik_z z^{(o)}} J_o(\gamma r^{(o)})/J_o(\gamma a)\cos(\varphi+\varphi_o) \quad (17)$$

$$\bar{E}^2 : 2\pi k \ddot{z}^{(2)} = \int_C dk_z b(k_z)(\gamma J_o'(\gamma r^{(o)}) r^{(1)} + ik_z J_o(\gamma r^{(o)}) z^{(1)}) e^{ik_z z^{(o)}} \cos(\varphi+\varphi_o) \quad (18)$$

The radial equations have been omitted. Equations (16) belong to the motion of a free particle. Equations (17) and (18) give the first, and second iteration, respectively. The kind of separation achieved renders the assumption of the series (15) reasonable. It would be ultimately justified, if one proved that the order of magnitude of $r^{(n)}$ and $z^{(n)}$ does not increase with ascending n. This has been shown up to n = 2.

When putting

$$r^{(o)} = \dot{r}_o\,\varphi + r_o \qquad z^{(o)} = \varphi/k \quad (19)$$

into (18) one has to expand with respect to $\dot{r}_o$, because otherwise one cannot integrate with respect to φ. Throughout we drop terms non-linear in $\dot{r}_o$. After we have integrated once or twice with respect to φ, we evaluate with respect to k_z by Cauchy's residue theorem. For $\varphi = kL > kg/2$ ($\varphi = -kL < kg/2$) (L = distance of the point of observation from the centre z = 0) one closes the path C by a semi-circle in the upper (lower) plane. $J_o(\gamma a)$ yields simple poles at $k_z = \pm i\eta_\rho$, the integrations with regard to φ poles of various orders at $k_z = \pm k$. The former yield terms proportional to $\exp(-\eta_\rho L)$. In general one considers only $L \to \infty$ and then solely the contributions due to the poles at $\pm k$ remain. These correspond to the singular behaviour of the Dirichlet-Integral in Refs. 1,2. If $|\varphi| < kg/2$ as for example, in the calculation of the S-coefficient, then one has to indent the path C at $k_z = 2\pi s/g$ ($s = 0, \pm 1, \pm 2, \ldots$) and to decompose $\sin(k_z g/2)$ in $b(k_z)$ into the exponentials. We may treat the integrals arising as above taking into account the simple poles at $k_z = 2\pi s/g$ which yield the first sum in $S(k_z,r)$, Table III.

Crossing the whole gap: The main formulae and results for $L \to \infty$ up to 1st order in $\bar{E}$ and $\dot{r}_o$ are exhibited in Table I. The longitudinal quantities completely agree withthose given earlier[1,2], but not the transversal ones; partly, because here the magnetic field is not included, partly due to some reasons intrinsic to the method employed in Ref. 2 which not yet have become entirely clear. In Table II an example is given for the formulae one finds for finite L. We choose the gain of energy between z = -L and z = +L. The series are not so frightening, the exponentials contained in each term decreasing rather rapidly.

First Half of the Gap: The energy gain and the phase jump $\Delta\varphi_1$ may be expressed by the Transit Time Factor T and the S-coefficient. In the derivation one partly uses Cauchy's residue theorem, partly transformations of integrals. Results for $L \to \infty$ are shown in Table III. They are in concordance with Ref. 2, apart from the impossibility of factoring $S(k,r) \neq S_o I_o(k_r r)$. It does not seem to be possible to express transverse motion by the S-coefficient defined in (12).

Second Order Contributions: So far we did not succeed in evaluating (18) in a satisfactory manner. To get rough estimates, $r^{(1)}$ and $z^{(1)}$ were replaced in (18) by $r^{(o)}$ and $z^{(o)}$ and the resulting expressions treated in the usual manner. The arising formulae let one hope that $r^{(2)}$ and $z^{(2)}$ are of about the same order of magnitude as $r^{(1)}$ and $z^{(1)}$ so that the powers of $\bar{E}$ in front of all terms in (15) really determine the magnitude.

5. Relativistic Motion

When treating the motion relativistically, the effects of the mass variation are of the same order of magnitude as those due to the magnetic field. It is possible to include the former too in a perturbation treatment. Integrating once the equations of motion we have (m = rest mass):

$$\vec{v}\,(1-\beta^2)^{-1/2} = \vec{v}_o\,(1-\beta_o)^{-1/2} + (e/(m\omega))\int_0^{\varphi} d\varphi\,(\vec{E} + \vec{v}\times\vec{B}) \qquad (20)$$

Squaring and solving the left side for β, we form thereafter:

$$(1-\beta^2)^{1/2} = (1-\beta_o^2)^{1/2}\Big[1 + 2E_c(1-\beta_o^2)^{1/2} \int d\varphi \left\{(\vec{\beta}_o, \vec{E}/E_1) + (\vec{\beta}_o, \vec{v}, \vec{B}/E_1)\right\} + E_c^2 \ldots\Big]^{1/2} \qquad (21)$$

$E_c = 3.6 \times 10^{-3}$ by the use of (1). Multiplying (20) by (21) we arrive at an equation $\vec{v} = \ldots$ The right-hand side is first expanded into powers of E_c, then the series (15) are introduced and we develop in powers of $\bar{E}$. There are now lots of terms, each accompagnied by a $E_c^n \bar{E}^m$. However, the order of magnitude of relativistic contributions is not determined by E_c, but by $\bar{E}\beta^2$ as clearly revealed in the results. Thus, it depends on β_o, which terms are relatively great and which can be neglected. Fig. 1 shows the relative magnitude of the leading powers in $\bar{E}$ and of $\bar{E}\beta^2$ in dependence on β and on energy. At 0.5 MeV the second iteration of the electric field ($\sim \bar{E}^2$) seems to contribute by about 10% while relativistic quantities are negligible. For 50 MeV the situation has completely reversed. All terms with the exception of the energy gain contain at least a factor $(1-\beta_o^2)^{1/2}$.

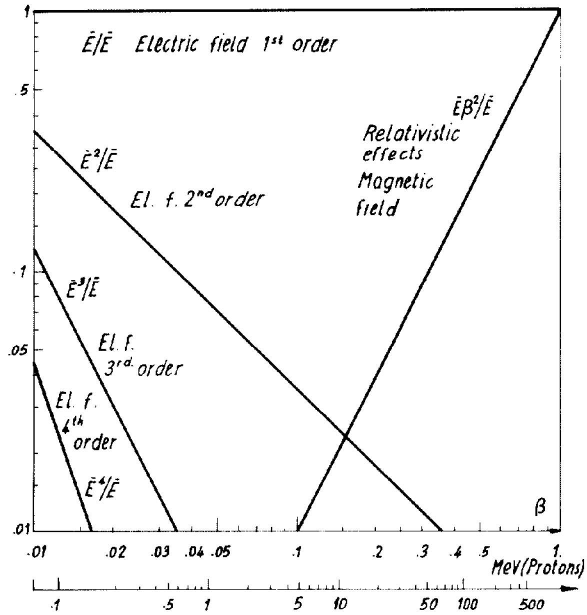

Fig. 1 Relative magnitude of parameters

The integrals are evaluated as in Section 4, retaining only 1st order terms in $\bar{E}$, E_c and r_o. In deriving the energy gain, one starts from the usual relativistic expression for the energy gain, uses (21) and ends up with:

$$W = e\int d\varphi\,(\dot{z}_o E_z + \dot{r}_o E_r) \qquad (22)$$

where $r^{(o)}$ and $z^{(o)}$ have to be inserted for r and z in the fields. The effect of the magnetic field vanishes to lowest order. Results are compiled in Table IV. But we believe that in the expression $(W_+ - W_-)_r$ the term in brackets coming from $\dot{r}_o E_r$, should be dropped as being of the order of r_o^2, though this is not clearly visible. Otherwise $(W_+ - W_-)_r$ would not go over into $(W_+ - W_-)$ for small β.

Conclusions

The formulae given in this paper and derived by first order perturbation theory seem to be sufficient (for say 3-4%) to treat beam dynamics in a linac gap as long as space charge effects and radiation losses can be neglected, for any energy of the protons greater than 5 MeV. At lower energies the 2nd order effects become important, at 0.5 MeV they could rise to 12% ; unfortunately there exist at present no expressions for these. Numerical investigations in simple cases show that, as far as energy gain is concerned, 1st order results are as accurate as 1% even at 0.5 MeV. But energy is stationary, while the other quantities are not.

I am very indebted to P.M. Lapostolle for the suggestion of this topic and for many helpful discussions.

References

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Table I. Non-relativistic change of energy, phase, transversal velocity and position across a gap.

$$W_+ - W_- = \quad = eV_o \; T_o I_o \; \cos\varphi + eV_o \, d/dk(T_o k_r I_1)\, r' \sin\varphi$$

$$\varphi_+ - \varphi_- = \varphi(kL) - \varphi(-kL) - L\dot{z}^{-1}(kL) + L\dot{z}^{-1}(-kL) \quad = \alpha k \, d/dk(T_o I_o) \; \sin\varphi - \alpha k \, d^2/dk^2(T_o k_r I_1)\, r' \cos\varphi$$

$$r'_+ - r'_- = \quad = -\alpha \; T_o k I_1/k_r \; \sin\varphi + \alpha \, d/dk(T_o k \, I'_1) \; r' \cos\varphi$$

$$r_+ - r_- = r(kL) - kL(-kL) - kL(\dot{r}(kL) + \dot{r}(-kL)) \quad = -\alpha \, d/dk(T_o k I_1/k_r) \cos\varphi - \alpha \, d^2/dk^2(T_o k \, I'_1)\, r' \sin\varphi$$

Table II. Non-relativistic energy change between finite observation points.

$$W(kL) - W(-kL) = m\omega^2 \dot{z}_o E \, (\dot{z}^{(1)}(kL) - \dot{z}^{(1)}(-kL)) = W_+ - W_- + eE_1/a^2 \sum_\rho J_o(j_\rho r/a)\,[B_\rho(kL) - B_\rho(-kL)]$$

$$-eE_1 r'/a^3 \sum_\rho j_\rho J_1(j_\rho r/a)\,[LB_\rho(kL) + LB_\rho(-kL) - C_\rho(kL) + C_\rho(-kL)]$$

$$B_\rho(\pm kL) = b(i\eta_\rho)\, e^{-\eta_\rho L}\, j_\rho \,[k \sin(\varphi_o \pm kL) \mp \eta_\rho \, \sin(\varphi_o \pm kL)]/[J_1(j_\rho)\eta_\rho(\eta_\rho^2 + k^2)]$$

$$C_\rho(\pm kL) = b(i\eta_\rho)\, e^{-\eta_\rho L}\, j_\rho \,[(\eta_\rho^2 - k^2)\cos(\varphi_o \pm kL) + 2\eta_\rho k \sin(\varphi_o \pm kL)]/[J_1(j_\rho)\eta_\rho(\eta_\rho^2 + k^2)^2]$$

$$b(i\eta_\rho) = 2g^2 \eta_\rho \sinh(\eta_\rho g/2) \sum_s B_s (-1)^s (\eta_\rho^2 g^2 + (2\pi s)^2)^{-1} \qquad \eta_\rho = (j_\rho^2/a^2 - k_o^2)^{1/2}$$

Table III. The S-coefficient. Change of energy and phase in the first half of a gap.

$$S(k_z, r) = T_o(k_z)\, \mathrm{ctg}(k_z g/2)\, I_o(k_r r) -$$

$$- J_o(k_o a)\left[2 \sum_s B_s \frac{g k_z}{(2\pi s)^2 - (g k_z)^2} \times \frac{I_o(r[(2\pi s/g)^2 - k_o^2]^{1/2})}{I_o(a[(2\pi s/g)^2 - k_o^2]^{1/2})} + \frac{4 k_z g}{a^2} \sum_\rho \frac{J_o(j_\rho r/a)}{J_1(j_\rho)} \times \frac{e^{-\eta_\rho g/2}}{\eta_\rho^2 + k_z^2} \times \right.$$

$$\left. \times \sum_s \frac{(-1)^s B_s}{\eta_\rho^2 g^2 + (2\pi s)^2} \right]$$

$$\Delta\varphi_1 = \varphi^{(1)}(0) - \varphi^{(1)}(-kL) - L/\dot{z}^{(1)}(-kL) \qquad = \bar{E}k\,[z^{(1)}(-kL) - kL\,\dot{z}^{(1)}(-kL)]$$

$$= (\varphi_+ - \varphi_-)/2 - \alpha k/2\,[dS/dk \, \cos\varphi + d^2/dk^2(\partial S/\partial r_o)\, r' \sin\varphi]$$

$$\Delta w_1 = (W_+ - W_-)/2 + eV_o/2\,[\; S \, \sin\varphi - \, d/dk\,(\partial S/\partial r_o)\, r' \sin\varphi]$$

Table IV. Relativistic change of energy and phase, transversal velocity and phase across a gap.

$$(W_+ - W_-)_r = (W_+ - W_-) \quad [\, - eV_o \quad T_o k \, I_1/k_r \quad r' \sin\varphi]$$

$$(\varphi_+ - \varphi_-)_r (1 - \beta_o^2)^{-1/2} = (\varphi_+ - \varphi_-)(1 - k_o^2/k^2) + \alpha k \, k_o^2/k^2 \quad T_o k \, I_1/k_r \quad r' \cos\varphi$$

$$(r'_+ - r'_-)_r (1 - \beta_o)^{-1/2} = (r'_+ - r'_-)(1 - k_o^2/k^2) + \alpha \, k_o^2/k^2 \quad T_o\,(I'_1 - I_o) \quad r' \cos\varphi$$

$$(r_+ - r_-)_r (1 - \beta_o^2)^{-1/2} = (r_+ - r_-)(1 - k_o^2/k^2) + \alpha \, k_o^2/k^2\,[d/dk(T_o I_1) - T_o I_1/k - d/dk(T_o I_o)]\; r' \sin\varphi$$

Common to all tables:

$$\alpha = eV_o/(2W) \qquad k_o^2/k^2 = (dz/dt)_o^2/c^2 \approx \beta_o^2$$

$$W = m/2\,(dz/dt)_o^2 = m/2\,\dot{z}_o^2\,\omega^2 \qquad m = \text{rest mass} \qquad 1 - k_o^2/k^2 = k_r^2/k^2 \approx 1 - \beta_o^2$$

The argument $k = \omega/(dz/dt)_o$ of $T_o(k)$, $k_r(k) r_o$ of $I_n(k_r r)$ and k, r_o of $S(k, r_o)$ and the subscript $_o$ of φ_o, r_o and $r'_o = dr_o/dz$ have been dropped. $s = -\infty \,..\, -1, 0, 1 \,..\, +\infty$; $\rho = 1, 2, .. \infty$