# ORBITS IN FIXED FIELD ALTERNATING GRADIENT SYNCHROTRON*

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## Summary

An analytical investigation has been made of the motion of a charged particle in an axially symmetric steady magnetic field. For a given momentum, expressions for the shape of the equilibrium orbit and betatron oscillation frequencies have been derived in terms of the field index k, the spiral parameter K, and the set of field coefficients $g_{nm}$. These algebraic results have been programmed for the IBM 704 computer. For a 500 MeV fixed field alternating gradient (FFAG) synchrotron designed as an injector,[1] computations yield radial and vertical tunes averaged with respect to momentum of $\nu_r = 3.23$, $\nu_z = 2.37$ with variations of $|\Delta\nu_r| \leq 0.0001$, $|\Delta\nu_z| \leq 0.0004$ depending on the exact momentum. For comparison, Runge-Kutta evaluations which require a factor of six longer computing times yield average tunes of $\nu_r = 3.22$ and $\nu_z = 2.49$.

## Introduction

The structure of radial straight sections in a spiral sector FFAG accelerator has not been treated analytically in sufficient detail to yield quantitative conclusions starting from a set of general magnetic field coefficients. For the complicate field, the method of its expansion is more important than the expansion of the equations of motion. It is found that by assuming the solution of an equilibrium orbit in a median plane to be a non-linear form

$$R = R_q \exp\left[r(\theta)\right] \qquad \text{with} \int_0^{2\pi} r(\theta)\, d\theta = 0\,, \tag{1}$$

instead of the linear form[2]

$$R = R_{av}(1 + r)\,,$$

the field can be systematically expanded around $R_q$ and have convergent series. The number of betatron oscillations per revolution $\nu_r$ and $\nu_z$ for radial and vertical motion varies periodically with particle momentum. Their deviations from constant parts are expressed in terms of the field coefficients. The reason that Cole and Morton[3] did not obtain the deviations and concluded that $\nu_r$ and $\nu_z$ were independent of momentum is explained.

## Form of the Magnetic Field

Let N and M be the number of spiral sectors and radial straight sections, respectively. On the median plane, the field in cylindrical coordinates is taken to be

$$H_z = H_o\left(\frac{R}{R_o}\right)^k \sum_{n,m} g_{nm} \exp\left[in\left(K \ln \frac{R}{R_o} - N\theta\right) + imM\theta\right], \tag{2}$$

where k is the field index; K, the spiral parameter; $R_o$, an arbitrary reference radius; and $g_{nm}$, the set of field coefficients. The convergence of the series requires that $g_{nm}$ approaches zero as $|n|$ or $|m|$ approaches infinity. $H_z$ is real. This reality condition leads to

$$g_{nm} = g^*_{-n,-m} \quad \text{and} \quad g_{n,-m} = g^*_{-n,m}\,.$$

Let

$$g_{nm} = \left(a_{nm} - d_{nm}\right) - i\left(b_{nm} + c_{nm}\right),$$
$$g_{n,-m} = \left(a_{nm} + d_{nm}\right) + i\left(b_{nm} - c_{nm}\right).$$

The field can be expressed as

$$H_o\left(\frac{R}{R_o}\right)^k \sum_{\substack{n\geq 0\\ m\geq 0}} f_{nm}\Bigg[a_{nm} \cos n\left(K \ln \frac{R}{R_o} - N\theta\right) \cos mM\theta$$
$$+ b_{nm} \cos\left(K \ln \frac{R}{R_o} - N\theta\right) \sin mM\theta$$
$$+ c_{nm} \sin\left(K \ln \frac{R}{R_o} - N\theta\right) \cos mM\theta$$
$$+ d_{nm} \sin\left(K \ln \frac{R}{R_o} - N\theta\right) \sin mM\theta\Bigg]\,, \tag{3}$$

where

$$f_{nm} = 4, \quad f_{om} = f_{no} = 2, \quad \text{and } f_{oo} = 1\,.$$

*Work performed under AEC Sub-Contract from Argonne National Laboratory

**Now at Argonne National Laboratory, Particle Accelerator Division

## Equation of Motion

For the motion of a charged particle in a magnetostatic field, the Euler-Lagrange equations give

$$\frac{d}{d\theta}\frac{R'}{\sqrt{R'^2+R^2+Z'^2}} - \frac{R}{\sqrt{R'^2+R^2+Z'^2}}$$

$$= \frac{e}{pc}\left(RH_z - Z'H_\theta\right), \qquad (4)$$

$$\frac{d}{d\theta}\frac{Z'}{\sqrt{R'^2+R^2+Z'^2}} = \frac{e}{pc}\left(R'H_\theta - RH_r\right), \qquad (5)$$

where

$$R' = \frac{dR}{d\theta}, \qquad Z' = \frac{dZ}{d\theta}.$$

The magnetic field in a circular accelerator is so designed that there exists a symmetry plane (median plane), and everywhere on this plane the field $H_z$ is perpendicular to the plane. The field nearby the plane can be adequately specified on the plane. From $\nabla \times H = 0$, the fields off the plane $H_z$ and $H_r$ are

$$H_\theta \simeq \frac{\partial H_\theta}{\partial z} Z = \frac{1}{R}\frac{\partial H_z}{\partial \theta} Z,$$

$$H_r \simeq \frac{\partial H_r}{\partial z} Z = \frac{\partial H_z}{\partial R} Z.$$

Then Eqs. (4) and (5) become

$$\frac{d}{d\theta}\frac{R'}{\sqrt{R'^2+R^2}} - \frac{R}{\sqrt{R'^2+R^2}} = \frac{e}{pc} RH_z \qquad (6)$$

$$\frac{d}{d\theta}\frac{Z'}{\sqrt{R'^2+R^2+Z'^2}} = \frac{e}{pc}\left(\frac{R'}{R}\frac{\partial H_z}{\partial\theta} - R\frac{\partial H_z}{\partial R}\right) Z. \qquad (7)$$

Let the solution for an equilibrium orbit in Eq. (6) be of the form in Eq. (1). It will turn out that $R_q$ is very nearly the average radius of the equilibrium orbit. Let a dimensionless field be

$$h = \frac{H_z}{H_o}\left(\frac{R_o}{R_q}\right)^k .$$

Then Eq. (6) becomes

$$\frac{d}{d\theta}\frac{r'}{\sqrt{r'^2+1}} - \frac{1}{\sqrt{r'^2+1}} = -\lambda h \exp(r), \qquad (8)$$

where

$$r' = \frac{dr}{d\theta}, \quad \text{and } \lambda = -\frac{eH_o}{pc}\left(\frac{R_q}{R_o}\right)^k .$$

h and $\lambda$ are dimensionless quantities and for a normalized field their magnitudes are equal to 1.

To study the small oscillations in radial or vertical direction, one must expand the equation of motion (6) or (7) about the equilibrium orbit. Let the solution of the oscillation be

$$R = R_q (1 + x)\exp(r), \quad \text{or} \quad Z = R_q y \exp(r).$$

Linearizing the differential equation (6) or (7) in x or y, the radial or vertical oscillation obeys

$$x'' + r'x' = -\lambda\left(h + \frac{\partial h}{\partial r}\right) x \exp(r), \text{ or} \qquad (9)$$

$$y'' + r'' = -\lambda\left(r'\frac{\partial h}{\partial\theta} - \frac{\partial h}{\partial r}\right) y \exp(r). \qquad (10)$$

## Equilibrium Orbit

The function h exp(r) has a Taylor's series expansion in powers of r; i.e.,

$$\sum_{n,m}\left[1 + r(k+1+inK) + \frac{r^2}{2}(k+1+inK)^2 + \ldots\right]$$

$$\times\ g_{nm}\exp\left[inKr_f + i(-nN + mM)\theta\right],$$

where

$$r_f = \ln\frac{R_q}{R_o}.$$

For the sake of simplicity, only the first few terms are considered here. The question is how many terms are necessary for the required accuracy. For the structure chosen as an example, $g_{oo}$ and $g_{\pm 1, m}$ are the dominant field coefficients. This can be justified from Table I. Therefore, the number of terms required in the expansion of h exp(r) for the accelerator depends on the value of

$$r\sqrt{(k+1)^2 + K^2} \qquad (11)$$

instead of

$r\sqrt{(k+1)^2 + (nK)^2}$.

If expression (11) is about 1, the error is about 10% when three terms are taken into account. For the specific structure,[1] it is about 0.1 ($r \sim 10^3$, $k = 8.2$, $K = 75$). Only two terms are needed and the error is less than 0.5%.

The number of periods of magnetic field per revolution, P, is the greatest common divisor of N and M. Then, $s = N/P$ is the number of spirals and $q = M/P$ is the number of radial straight sections per period of the magnetic field. The solution of Eq. (8) is assumed to be of the form

$$r = \sum_{\ell} r_{\ell} \exp(i\ell P\theta) \quad \text{with } r_o = 0 .$$

The important terms in the equation of motion (8) are

$$1 + \sum_{\ell, j} \left[(\ell P)^2 r_{\ell} - \lambda\left(A_{\ell} + r_j B_{\ell - j}\right)\right] \exp(i\ell P\theta) = 0 ,$$

where

$$A_{\ell} = \sum g_{nm} \exp\left(inKr_f\right) ,$$

$$B_{\ell} = \sum g_{nm} (k + 1 + inK) \exp\left(inKr_f\right) .$$

The summations are over n and m under the condition that $-sn + qm = \ell$. Comparing the coefficients for the harmonic number $\ell$, $\lambda$ and $r_{\ell}$ can be given in terms of k, K and $g_{nm}$. Let

$$\lambda = \lambda^{(0)} + \lambda^{(1)} + \dots ,$$

$$r_{\ell} = r_{\ell}^{(0)} + r_{\ell}^{(1)} + \dots ,$$

where the superscript indicates the order of approximation. In the zero approximation, $\lambda$ and $r_{\ell}$ are

$$\lambda^{(0)} = 1/A_o, \text{ and}$$

$$r^{(0)} = \lambda B_{\ell} \Big/ \left[\ell^2 P^2 - \lambda(k+1)\right] .$$

There are three pairs (n, m) which are important for the condition $-sn + qm = 0$. These are (0, 0), (q, s), and (-q, -s). $\lambda^{(0)}$ becomes

$$\lambda^{(0)} \simeq \frac{1}{g_{oo}} - \frac{2\sqrt{g_{qs}^* g_{qs}}}{g_{oo}} \cos\left(qKr_f - a_{qs}\right) ,$$

where $a_{qs}$ is the argument of $g_{qs}$. The magnetic field is scaled for a finite number of points in the radial direction, and its shape repeats itself q time in a superperiod. Therefore, $\lambda$ is a periodic function of the radius R or momentum p. $\lambda^{(0)}$ can be separated into two parts: the part due to the scaling field $\lambda_s^{(0)}$, and the deviation due to radial cuts in spiral geometry, $\Delta\lambda$. We get

$$\lambda_s^{(0)} = 1/g_{oo} .$$

The maximum deviation is

$$\Delta\lambda_{max} = 2\sqrt{g_{qs}^* g_{qs}} \Big/ g_{oo}^2 .$$

To the first order approximation and taking the value of $r_{\ell}^{(0)}$ for $r_{\ell}$, $\lambda^{(1)}$ and $r_{\ell}^{(1)}$ are

$$\lambda^{(1)} = -\frac{(k+1)}{g_{oo}^2} S_o , \text{ and}$$

$$r_{\ell}^{(1)} = \frac{\lambda^2}{\ell^2 P^2 - \lambda^{(0)}(k+1)} S_o ,$$

where

$$S_o = \sum \frac{f_{nm}\left[a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right]}{\left[(-nN + mM)^2 - \lambda^{(0)}(k+1)\right]} .$$

The summation is for $n \geq 0$, $m \geq 0$, except both $n = 0$ and $m = 0$.

## Betatron Oscillation Frequencies

Knowing the equilibrium orbit, the betatron oscillations can be studied from Eqs. (9) and (10). In the second order differential Eq. (10), the term r" corresponds to the main driving force which will cause vertical instability. The contribution of the term to the vertical oscillation frequency is small and can be neglected. The term r'x' in Eq. (9) can also be neglected. The error is about 1%. Substituting the dimensionless field h into Eqs. (9) and (10) and expanding $\exp\left[r(k + 1 + inK)\right]$ in the power of r, we get

$$x'' + \lambda x \sum_{\ell, j} \left[B_{\ell} + r_j C_{\ell - j}\right] \exp(i\ell P\theta) = 0 ,$$

$$y'' + \lambda y \sum_{\ell, j} \left[A_{\ell} - B_{\ell} + r_j\left(B_{\ell - j} - C_{\ell - j} - jPD_{\ell - j}\right)\right]$$

$$\times \exp(i\ell P\theta) = 0 ,$$

where

$$C_\ell = \sum g_{nm} (k + 1 + inK)^2 \exp\left[inKr_f\right] ,$$

$$D_\ell = \sum g_{nm} (-nN + mM) \exp\left[inKr_f\right] .$$

The summations are over n and m under the condition $-sn + qm = \ell$. These are the Floquet-type equations. The values of radial and vertical betatron frequencies $\nu_r$ and $\nu_z$ can be obtained by the application of quantum second order perturbation theory.[2]

As with the dimensionless quantity $\lambda$, $\nu_r$ and $\nu_z$ are not constant. Let the parts due to the scaling field be $\nu_{r,s}$ and $\nu_{z,s}$, and the deviations due to the radial cuts be $\Delta\nu_r$ and $\Delta\nu_z$. The values of $\nu_{r,s}$ and $\nu_{z,s}$ turn out to be the averaged values of $\nu_r$ and $\nu_z$, respectively. Let

$$S_1 = \sum \frac{f_{nm}\left[(nK)^2 - (k+1)^2\right]\left[a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right]}{\left[(-nN + mM)^2 - \lambda^{(0)}(k+1)\right]} ,$$

$$S_2 = \sum \frac{f_{nm}\left[(nK)^2 - k(k+1)\right]\left[a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right]}{\left[(-nN + mM)^2 - \lambda^{(0)}(k+1)\right]} ,$$

$$S_3 = \sum \frac{f_{nm}\left[(-nN + mM)^2\right]\left[a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right]}{\left[(-nN + mM)^2 - \lambda^{(0)}(k+1)\right]} ,$$

$$S_r = \sum \frac{f_{nm}\left[(nK)^2 + (k+1)^2\right]\left[a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right]}{\left[(-nN + mM)^2 - 4\nu_{r,s}^2\right]} ,$$

$$S_z = \sum \frac{f_{nm}\left[(nK)^2 + k^2\right]\left[a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right]}{\left[(-nN + mM)^2 - 4\nu_{z,s}^2\right]} .$$

The summations are for $n \geq 0$, $m \geq 0$, except both $n = 0$ and $m = 0$. Using the same method for simplifying $\lambda^{(1)}$ and $r_\ell^{(1)}$, the tunes due to the scaling field are

$$\nu_{r,s}^2 = k + 1 + \lambda^{(0)^2} \left[-S_1 + S_r\right] , \text{ and} \tag{12}$$

$$\nu_{z,s}^2 = -k + \lambda^{(0)^2} \left[S_2 + S_3 + S_z\right] . \tag{13}$$

Equations (12) and (13) are valid when

$$\frac{\sqrt{\left[(nK)^2 + (k+1)^2\right]\left[a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2\right]}}{\left[(-nN + mM)\right]^2} << 1 , \tag{14}$$

and

$$0 \leq \nu_{r,s} < \frac{P}{2} , \qquad 0 \leq \nu_{z,s} < \frac{P}{2} .$$

The maximum deviations $\Delta\nu_r$ and $\Delta\nu_z$ become

$$\Delta\nu_r = \frac{g_{oo}\Delta\lambda_{max}}{\nu_{r,s}} (k+2) + \frac{1}{g_{oo}^2} \left[-S_1 + S_r\right] \tag{15}$$

$$\Delta\nu_z = \frac{g_{oo}\Delta\lambda_{max}}{\nu_{z,s}} (k+1) + \frac{1}{g_{oo}^2} \left[S_2 + S_3 + S_z\right] . \tag{16}$$

Cole and Morton[3] neglected all the harmonics n higher than q/2. In this case, the term $\cos\left[qKr_f - a_{qs}\right]$ disappears or $\Delta\lambda$ is zero. This explains why they concluded that $\nu_r$ and $\nu_z$ were independent of momentum. Therefore, the harmonic number to be considered must be higher than the numbers of q and s.

## 500 MeV FFAG Synchrotron

It is of interest to consider the performance of a high intensity FFAG with a view to its possible use as an injector. A 500 MeV FFAG synchrotron has been studied.[1,4] The parameters of the accelerator related to the calculation of orbits and operation points are: N = 16, M = 72, P = 8, k = 8.2, K = 75, s = 2, and q = 9. The magnetic field repeats after a period of two spiral sectors or nine radial straight sections. On the median plane, the magnetic field of the accelerator is of the form in expression (3). The field coefficients $a_{nm}$, $b_{nm}$, $c_{nm}$, and $d_{nm}$ shown in Table I[5] are obtained from a computer program (MURA F46).

Using the expressions in the previous sections, the dynamic quantities are: $\lambda^{(0)} = 0.84$, $\lambda^{(1)} = -6 \times 10^{-3}$, $\Delta\lambda_{max} = 3.7 \times 10^{-5}$, $r_1^{(0)} = 3.5 \times 10^{-4}$, $r_2^{(0)} = 1.4 \times 10^{-3}$, . . . . $r_2$ is the highest value among $r_\ell$ since the main contribution is $a_{10}$ which is the highest coefficient besides $a_{00}$. The condition (14) is satisfied since for the pair (1,0), the value is about 0.15. For any other pairs (n, m), the values are much less than 0.1. Equations (12) and (13) have been programmed for the IBM 704 computer and for the

| | m \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_{nm}$ | 0 | 1.19228 | -.41445 | -.02562 | -.00160 | -.00599 | .00267 | .00017 | -.00028 | .00003 | -.00005 |
| | 1 | -.20740 | .09537 | .00805 | .00046 | .00214 | -.00113 | -.00006 | .00014 | -.00001 | .00002 |
| | 2 | -.07438 | .03517 | .00164 | .00062 | .00101 | -.00045 | -.00002 | .00002 | -.00001 | .00002 |
| | 3 | -.00345 | .00140 | .00008 | -.00001 | .00001 | .00010 | -.00001 | -.00001 | -.00000 | -.00001 |
| | 4 | .00816 | -.00433 | -.00004 | -.00020 | -.00022 | .00019 | .00000 | -.00001 | .00000 | -.00001 |
| $b_{nm}$ | 0 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 |
| | 1 | .02613 | -.00846 | .00116 | -.00063 | -.00034 | .00008 | -.00002 | .00001 | .00000 | -.00000 |
| | 2 | .00126 | .00180 | .00248 | -.00082 | -.00022 | -.00003 | -.00004 | .00003 | .00000 | .00000 |
| | 3 | -.00126 | .00103 | .00036 | -.00005 | .00004 | -.00002 | -.00000 | -.00000 | .00000 | .00000 |
| | 4 | -.00028 | -.00014 | -.00029 | .00024 | .00006 | .00000 | .00002 | -.00002 | -.00000 | .00000 |
| $c_{nm}$ | 0 | .00000 | -.14867 | -.08327 | .02047 | .00401 | .00012 | .00065 | -.00052 | -.00005 | .00006 |
| | 1 | .00000 | .03570 | .02445 | -.00746 | -.00159 | -.00002 | -.00026 | .00022 | .00001 | -.00003 |
| | 2 | .00000 | .01616 | .00917 | -.00325 | -.00047 | -.00007 | -.00012 | .00009 | .00001 | -.00001 |
| | 3 | .00000 | .00068 | .00042 | .00021 | .00002 | -.00000 | .00000 | -.00003 | .00000 | .00000 |
| | 4 | .00000 | -.00229 | -.00108 | .00083 | .00008 | .00002 | .00004 | -.00004 | -.00000 | .00000 |
| $d_{nm}$ | 0 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 | .00000 |
| | 1 | .00000 | -.00648 | -.00258 | .00058 | -.00010 | .00010 | .00002 | -.00003 | -.00000 | -.00000 |
| | 2 | .00000 | -.00462 | -.00058 | -.00011 | -.00037 | .00011 | .00001 | -.00001 | .00000 | -.00000 |
| | 3 | .00000 | -.00022 | .00009 | -.00015 | -.00005 | -.00001 | -.00000 | .00001 | -.00000 | -.00000 |
| | 4 | .00000 | .00073 | .00006 | .00001 | .00008 | -.00006 | -.00001 | .00000 | -.00000 | .00000 |

Table I. Median Plane Coefficients

structure yield the constant part of tunes:

$\nu_{r,s} = 3.23$, and $\nu_{z,s} = 2.37$.

For comparison, Runge-Kutta evaluations which require a factor of six longer computing times yield average tunes of

$\nu_r = 3.22$, and $\nu_z = 2.49$.

From Eqs. (15) and (16) the deviations are

$|\Delta\nu_r| \leq 0.0001$, and $|\Delta\nu_z| \leq 0.0004$.


## Acknowledgement

The author would like to thank Dr. S. C. Snowdon for his encouragement and helpful discussion.


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