A METHOD OF COMPUTING GENERALIZED ORBIT CONSTANTS IN SYNCHROTRON LATTICES*

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Collins[1] presented a method of graphical evaluation of certain orbit constants for alternating-gradient synchrotron lattices of the FODO type. His paper does not describe the process for obtaining the constants, which appears to have been by a combination of exact numerical solutions and analytic approximations. The present paper describes a process suitable for an electronic computer, and presents orbit constants for FODO and several other lattices: FOFDOD, FDO and the mirror pair DFFDO and FDDFO. In all cases, the results apply to simplified lattices with no drift spaces other than indicated by the "O" and with identical focusing and defocusing magnets. End effects are ignored, i.e., the magnets are sector magnets curved to match the synchrotron ring curvature, with end faces normal to the center line.

1. Results

The quantities obtainable from the graphs are β_{max} , β_{min} , X_o and α , where β is the betatron oscillation amplitude function, X_o is the characteristic length of the magnetic field gradient and α is the momentum compaction parameter = $\Delta C/C \div \Delta p/p$. By definition $X_o = B_o/G$, where G is the gradient and B_o is the center line field strength. The graphs give the quantities A_β , A_x and A_α as a function of μ and σ , from which one obtains β , X_o and α via the equations

$$\beta = A_\beta \, C/N \tag{1}$$

$$X_o = A_x \, C/N^2 \tag{2}$$

$$\alpha = A_\alpha/\nu^2 \quad . \tag{3}$$

Here C is the machine circumference, N is the number of periods, μ is the phase shift per period, ν is the number of oscillations per revolution, and σ is the fraction of C in straight sections. Depending on the type of lattice, β_{min} , β_{max} , X_o and α may or may not be functions of σ . Where they are, they are presented as a family of curves corresponding to σ = 0.25, 0.35, 0.45 and 0.55.

In the usual case of parameter juggling, C is fixed and B_o and σ are related by

$$B_o C \, (1 - \sigma) = 2\pi \, pc/e \tag{4}$$

where pc/e is the magnetic rigidity of the particle in units consistent with the product $B_o C$. The field at a distance r from center line is given by

$$B = B_o \left(1 \pm \frac{r}{X_o} \right) \quad . \tag{5}$$

2. Method of Computing the Parameters

All dimensioned variables are made dimensionless by the factor $1/\sqrt{K}$ which has the dimension of length, where $K = G/(pc/e)$ is the magnet force constant. If M is the matrix for a lattice element, let $\overline{M}$ be the corresponding dimensionless matrix. Thus the dimensionless matrix for a focusing magnet is, for example

$$\overline{F} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}$$

where $\psi = \sqrt{K}\, L$ and L is the magnet length. It is easily shown that $\overline{(M_1 \, M_2)} = \overline{M}_1 \, \overline{M}_2$ and that if V is a vector, $\overline{(MV)} = \overline{M} \, \overline{V}$. One can therefore multiply a series of dimensionless matrices or vectors and regenerate the dimensioned form from the end result.

Arbitrary values are picked for σ and ψ and the various dimensionless lattice element matrices are multiplied together giving the dimensionless lattice matrix $\overline{C}$ having elements $\overline{C}_{ij}$. The phase shift μ is then obtainable from[2]

$$\cos\mu = \tfrac{1}{2}\,(C_{11} + C_{22}) = \tfrac{1}{2}\,(\overline{C}_{11} + \overline{C}_{22}) \quad .$$

From[2] $C_{12} = \beta \sin\mu = \overline{C}_{12}/\sqrt{K}$, one gets

$$A_\beta = \frac{1 - \sigma}{n\,\psi} \, \frac{\overline{C}_{12}}{\sin\mu} \tag{6}$$

where n is the number of magnets in the lattice.

Similarly, from $\psi^2 = KL^2 = 2\pi L/(n\,N\,X_o)$, one gets

$$A_x = \frac{2\pi\,(1 - \sigma)}{n^2\,\psi^2} \quad . \tag{7}$$

As a test to insure that β is indeed an extremum, one can also compute $a = \tfrac{1}{2}\,(\overline{C}_{11} - \overline{C}_{22})/\sin\mu$ which should be essentially zero. If the exact location of the extremum is not known, the extreme value of β can be computed from β at any point in the lattice element, using the fact that $K\beta + \gamma$ is constant within a lattice element. (γ is the third betatron oscillation parameter of Courant and Snyder.[2])

From $\beta\gamma = 1 + a^2$

and $a = 0$ at an extremum in β ,

$$K\beta_x + 1/\beta_x = K\beta + (1 + a^2)/\beta$$

where β_x is the extreme value of β and a and β pertain to any other point in the lattice element.

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The solution of this quadratic is

$$\beta_x = \frac{\beta}{2}\left[1 + \frac{1+a^2}{K\beta^2} + \sqrt{\left(1 + \frac{1+a^2}{K\beta^2}\right)^2 - 4/K\beta^2}\,\right] . \quad (8)$$

The proper sign must be used for K : plus in a focusing magnet, minus in a defocusing magnet. Conveniently, $K\beta^2 = (\bar{C}_{12}/\sin\mu)^2$. In a drift space, the proper form for (8) is

$$\beta_x = \beta/(1 + a^2) . \quad (9)$$

The momentum compaction may also be obtained using the dimensionless matrix technique and a method described by Livingood.[3] The equilibrium orbit of a particle of momentum p + dp is obtained by a method described by Courant[4] and the path length is computed along this orbit. The equilibrium orbit is obtained by offsetting each magnet an amount $\Delta Y = X_o\,\Delta p/p$, plus for defocusing magnets, minus for focusing magnets. For example, the effect of an offset focusing magnet on vector $\vec{Y}_1$ could be expressed as

$$\vec{Y}_2 = F(\vec{Y}_1 - \vec{\Delta Y}) + \vec{\Delta Y}$$

where $\vec{\Delta Y} = \begin{pmatrix} \Delta Y \\ 0 \end{pmatrix}$.

An arbitrary initial vector $\vec{X}_1$ is acted on by the altered lattice, producing a vector $\vec{X}_2$. Thus,

$$\vec{X}_2 - \vec{e} = C\,(\vec{X}_1 - \vec{e})$$

or $$\vec{e} = (1 - C)^{-1}\,(\vec{X}_2 - C\,\vec{X}_1)$$

where $\vec{e}$ is the equilibrium orbit vector. If $\vec{X}_1 = 0$,

$$\vec{e} = (1 - C)^{-1}\,\vec{X}_2 .$$

Having the equilibrium orbit, the path length in a magnet is given by

$$\int \left[(1 + Y/\rho_o)^2 + Y'^2\right]^{\frac{1}{2}} dx$$

and in a drift space by $(1 + Y'^2)^{\frac{1}{2}}\,\Delta x$, where Y is the radial displacement and x is the azimuthal displacement. To first order, the increase in path length in the lattice is simply

$$\Delta C = \frac{1}{\rho_o}\int Y\,dx$$

where the integral is over every magnet in the lattice, and ρ_o is the radius of curvature at the magnet center line:

$$\rho_o = pc/(e\,B_o) = nN\psi/(2\pi\sqrt{K}) .$$

The integral is evaluated numerically by multiplying $\vec{e}$ successively by the matrix for each element in the lattice. Each magnet matrix is divided into eight submatrices, thus obtaining Y at nine positions (including the beginning), and the integral is obtained using Simpson's 1/3 Rule.

Then

$$\frac{1}{\rho_o}\int Y dx = \frac{2\pi\sqrt{K}}{nN\psi}\int Y dx = \frac{2\pi}{nN\psi}\int \bar{Y} dx = \frac{2\pi}{nN\psi}\,\Delta x\,\Sigma\,\bar{Y}_i$$

where $\Delta x = (1 - \sigma)/(8n)$ C/N and $\Sigma\bar{Y}_i$ means the Simpson's Rule summation, ΔY in dimensionless form is $\Delta\bar{Y} = \Delta Y\sqrt{K}$, so $\Delta p/p = \Delta\bar{Y}/(\sqrt{K}\,X_o)$, and using $\nu = N\mu/2\pi$

$$A_\alpha = \frac{1 - \sigma}{8\,n}\left(\frac{\mu}{n\psi}\right)^2 \frac{\Sigma\bar{Y}_i}{\overline{\Delta Y}} \quad (10)$$

Acknowledgements

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References

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3. J.J. Livingood, "Principles of Cyclic Particle Accelerators," (D. Van Nostrand, New York, 1961), p. 212.

4. E.D. Courant, "Computations of AGS Orbits with 704 Computer - I. Description of Computer Program," BNL Accelerator Dept. Internal Report EDC-36, January 7, 1960.

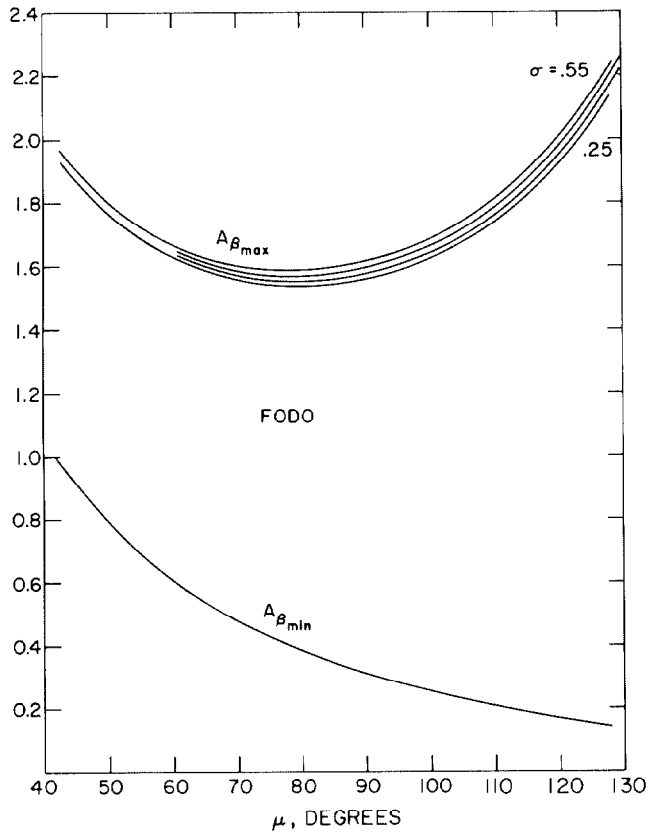

Fig. 1. $A_{\beta max}$ and $A_{\beta min}$ vs. phase shift per period μ, degrees, FODO lattice.

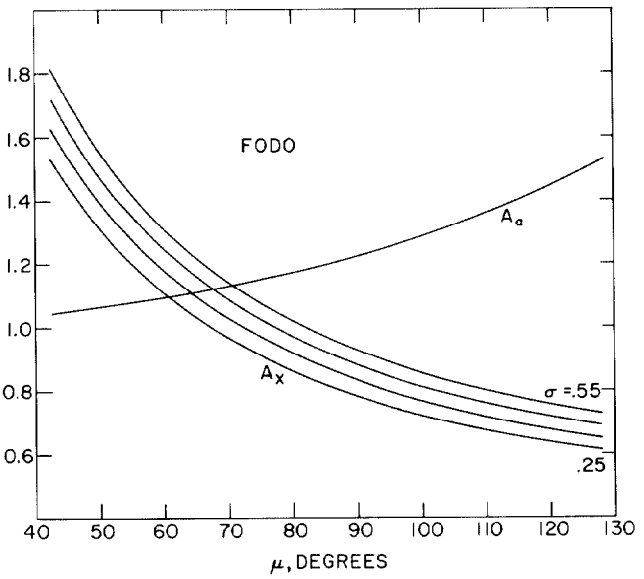

Fig. 2. A_X and A_α vs. μ, FODO lattice.

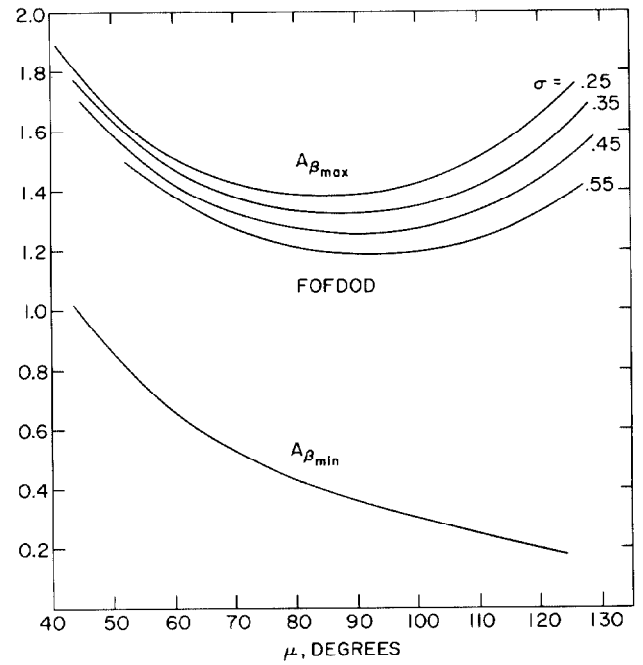

Fig. 3. $A_{\beta max}$ and $A_{\beta min}$ vs. μ, FOFDOD lattice.

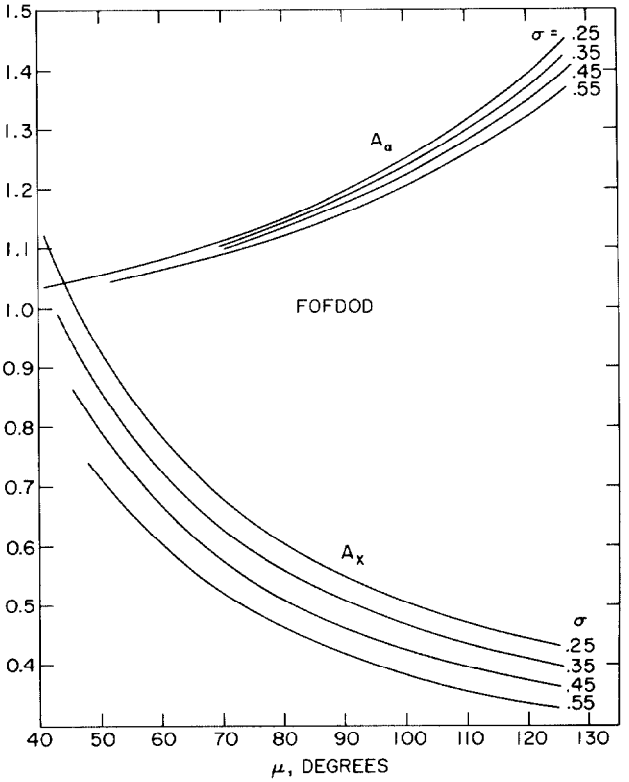

Fig. 4. A_X and A_α vs. μ, FOFDOD lattice.

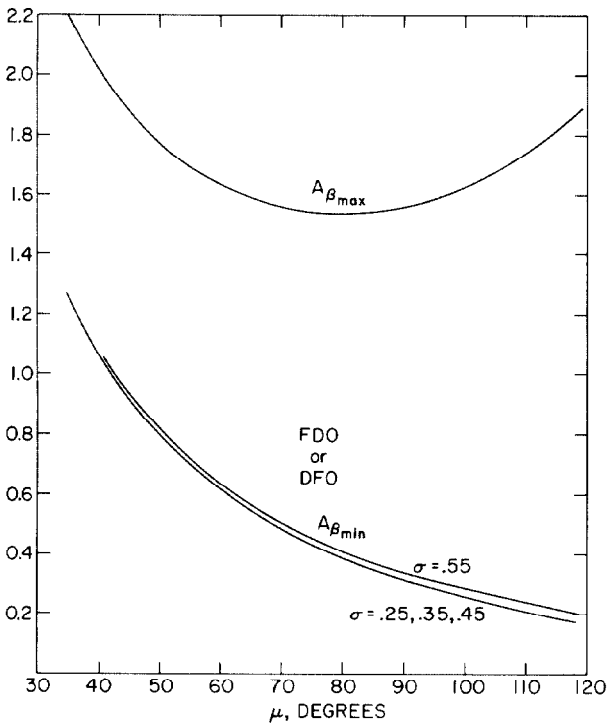

Fig. 5. $A_{\beta max}$ and $A_{\beta min}$ vs. μ, FDO or DFO lattice.

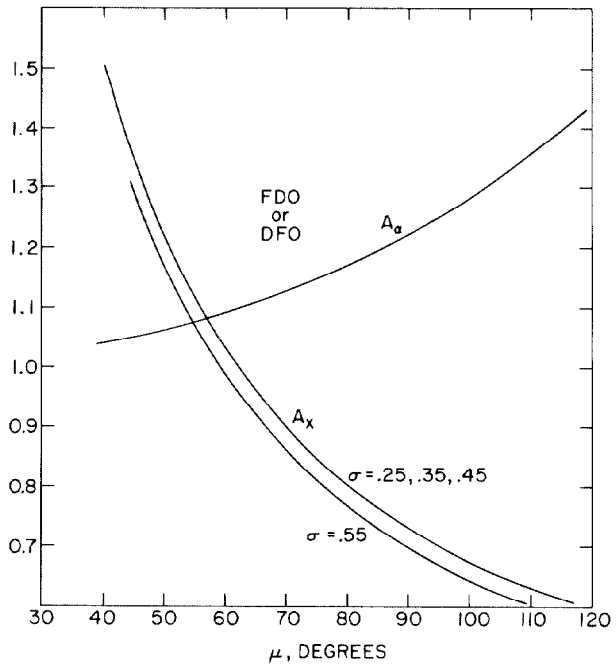

Fig. 6. A_x and A_α vs. μ, FDO or DFO lattice.

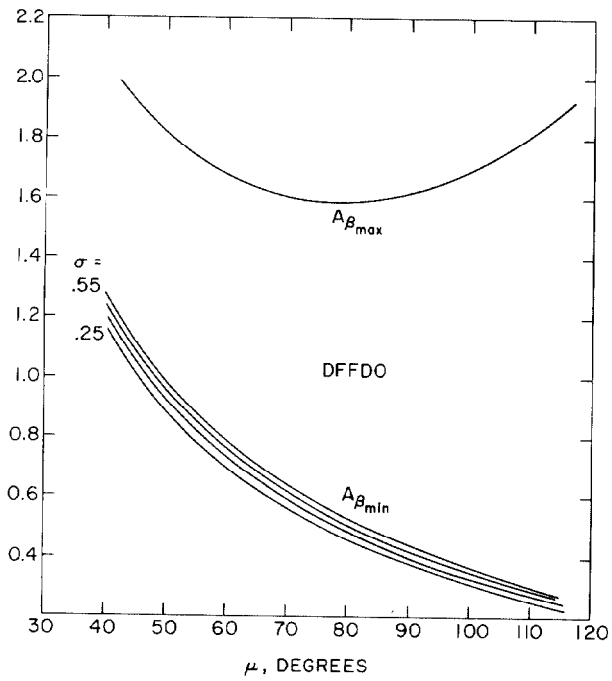

Fig. 7. $A_{\beta max}$ and $A_{\beta min}$ vs. μ, DFFDO lattice.

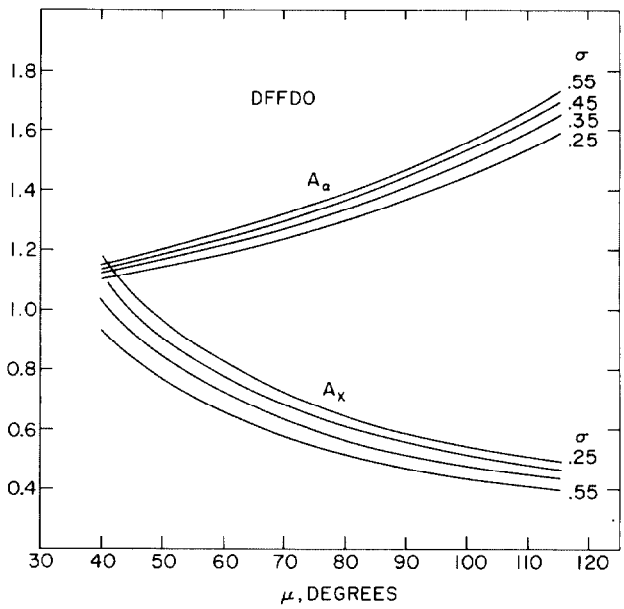

Fig. 8. A_x and A_α vs. μ, DFFDO lattice.

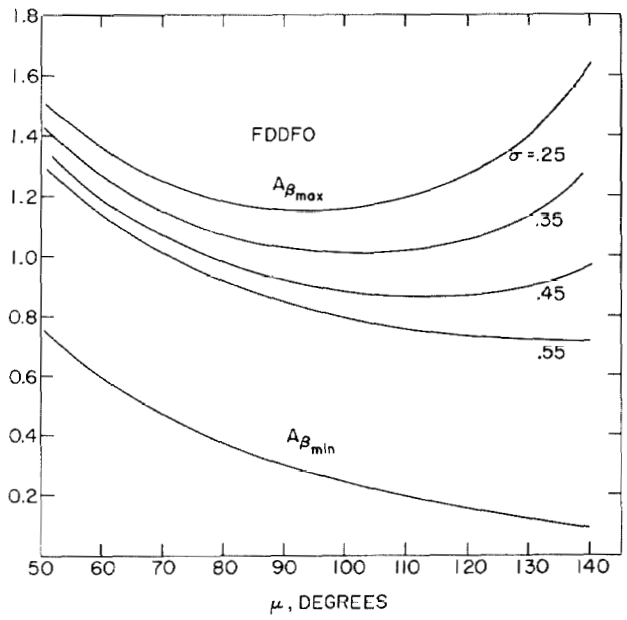

Fig. 9. $A_{\beta_{max}}$ and $A_{\beta_{min}}$ vs. μ, FDDFO lattice.

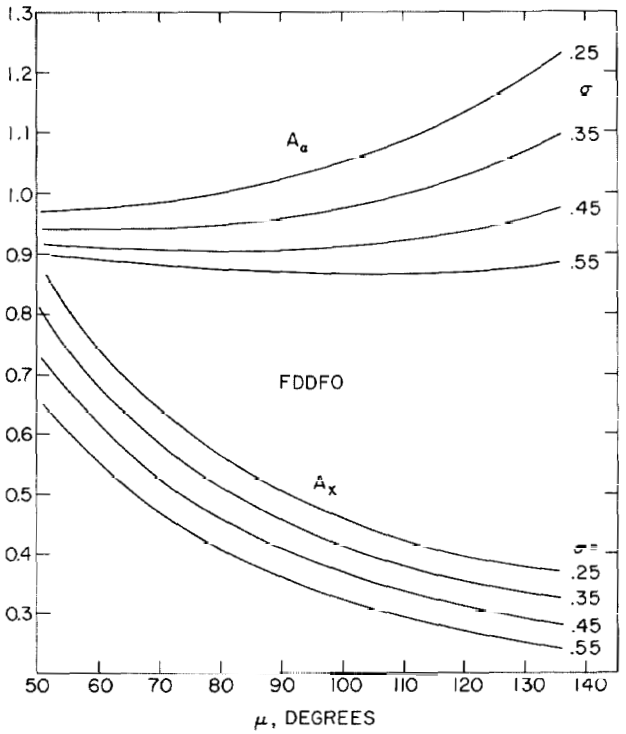

Fig. 10. A_X and A_α vs. μ, FDDFO lattice.