THREE DIMENSIONAL PROPERTIES OF MAGNETIC BEAM TRANSPORT ELEMENTS.*

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Summary. Analytical and experimental methods to describe precisely, in three dimensions, the magnetic fields of beam transport elements will be discussed. The influence of the fringing field on beam optical properties and magnet design will be presented. With appropriate boundary conditions the field components are expressed in terms of the magnetic potential to arbitrarily high orders with empirical coefficients and are grouped in a form to avoid the divergence of higher order terms. As an accurate description of the true field is obtained, a very precise trajectory can be computed for a known momentum. In addition, the more useful inverse problem of finding the momentum of a particle with known input and output trajectory coordinates is attacked. The trajectory solution is expressed as the sum of an analytical part plus perturbation terms. The first order solution for the momentum is easily obtained and the convergence of the subsequent perturbation calculations is very rapid even for extreme rays. The value of this method, with its precise treatment of fringing fields, is emphasized by treating in detail a wide angle spectrometer (120-in. wide, 36-in. long, 24-in. gap). The perturbation technique shows promise that an accurate instantaneous read-out of the momentum can be obtained by employing a modest number of perturbation corrections stored in the memory of the computer.

Multipole Description of Magnetic Field

The magnetic field of beam transport elements can be expressed in terms of the magnetic potential expanded in a series of multipolarities about the aperture centerline of each element. This discussion is restricted to elements in which the center of the aperture is a straight line, which is true for practically all beam transport lenses. Generally a median plane of mechanical symmetry occurs which also contains the centerline. This is a plane of magnetic anti-symmetry: the potential $\varphi(\theta) = \varphi(-\theta)$, where $\theta=0$ lies on the median plane, and the polar plane is normal to the Z-axis. The Z-axis coincides with the aperture centerline. For a quadrupole magnet a second plane orthogonal to the first and also containing the centerline is a plane of mechanical symmetry and magnetic anti-symmetry. For a dipole magnet, there is no necessity for this left-right mechanical symmetry, but when it does occur, as in the case of window frame dipoles, this plane is a plane of magnetic symmetry.

Let us now outline the potentials and fields for quadrupole magnet multipolarities and also for dipoles with left-right symmetry. C-type dipoles can be treated in this manner but their properties are not described in this paper.

[(1)] Let, $\varphi = \sum_{\lambda}\sum_{\mu} r^{\lambda} \sin \mu\theta f_{\lambda\mu}(z)$

be the magnetic potential expressed in cylindrical coordinates: where Z is the azimuthal axis, r and θ are the polar coordinates. There are no $\cos \mu\theta$ terms since $\varphi(-\theta) = -\varphi(\theta)$.

Substitute in $\nabla^2 \varphi=0$, apply boundary conditions, and group in powers of r. This reduces the number of indeterminate functions $f_{\lambda,\mu}(z)$.

The field components are easily obtained from the potential.

(i) For the circular quadrupole

$$[(2)]\quad B_r = \sin 2\theta\Big[\{2r\, f_{2,2}(z)\}+4r^3(-1/12)f^{ii}_{2,2}(z)+\dots\Big]$$
$$+\sin 6\theta\Big[\{6r^5 f_{6,6}(z)\}+8r^7(-1/28)f^{ii}_{6,6}(z)+\dots\Big]$$
$$+\sin 10\theta\Big[\{10r^9 f_{10,10}(z)\}+12r^{11}(-1/44)f^{ii}_{10,10}(z)+\dots\Big]$$
$$+\dots$$

$$[(3)]\quad B_\theta = \cos 2\theta\Big[\{2rf_{2,2}(z)\}+2r^3(-1/12)f^{ii}_{2,2}(z)+\dots\Big]$$
$$+\cos 6\theta\Big[\{6r^5 f_{6,6}(z)\}+6r^7(-1/28)f^{ii}_{6,6}(z)+\dots\Big]$$
$$+\cos 10\theta\Big[\{10r^9 f_{10,10}(z)\}+10r^{11}(-1/44)f^{ii}_{10,10}(z)+\dots\Big]$$
$$+\dots$$

$$[(4)]\quad B_z = \sin 2\theta\Big[r^2 f^{i}_{2,2}(z) + r^4(-1/12)f^{iii}_{2,2}(z)+\dots\Big]$$
$$+\sin 6\theta\Big[r^6 f^{i}_{6,6}(z) + r^8(-1/28)f^{iii}_{6,6}(z)+\dots\Big]$$
$$+\sin 10\theta\Big[r^{10} f^{i}_{10,10}(z)+ r^{12}(-1/44)f^{iii}_{10,10}(z)+\dots\Big]$$
$$+\dots$$

Inside the quadrupole where only two dimensional terms are found (i.e., $f_{n,n}(z)$ = constant), all derivatives with respect to z are zero and only the bracketed terms { } are present.

(ii) For the window frame dipole, the same procedure plus a transformation to rectangular coordinates gives for B_y, the main component of field, the expression:

$$[(5)]\quad B_y = f_{1,1}(z)+\{((-x^2-3y^2)/(8))\, f^{ii}_{1,1}(z)$$
$$+ (3x^2-3y^2)f_{3,3}(z)\}$$
$$+\{((x^4+6x^2y^2+5y^4)/(24 \times 8))f^{iv}_{1,1}(z)$$
$$+((-3x^4-6x^2y^2+5y^4)/(16))f^{ii}_{3,3}(z)$$

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$$+(5x^4-30x^2y^2+5y^4)f_{5,5}(z)\}$$

$$+\{((-x^6-9x^4y^2-15x^2y^4-7y^6)/(48\text{x}24\text{x}8))f^{vi}_{1,1}(z)$$

$$+((3x^6+15x^4y^2+5x^2y^4-7y^6)/(40\text{x}16))f^{iv}_{3,3}(z)$$

$$+((-5x^6+15x^4y^2+45x^2y^4-7y^6)/(24))f^{ii}_{5,5}(z)$$

$$+(7x^6-105x^4y^2+105x^2y^4-7y^6)f_{7,7}(z)\}$$

The equations for B_x and B_z are omitted: they will be expressed in a different form.

On the median plane (y=o)

$$[(6)]\ B_y = f_{1,1}(z)+[(-1/8)f^{ii}_{1,1}(z)+3f_{3,3}(z)]x^2$$
$$+[(1/24\text{x}8)f^{iv}_{1,1}(z)-(3/16)f^{ii}_{3,3}(z)+5f_{5,5}(z)]x^4$$
$$+[(-1/48\text{x}24\text{x}8)f^{vi}_{1,1}(z)+(3/40\text{x}16)f^{iv}_{3,3}(z)$$
$$+(-5/24)f^{ii}_{5,5}(z)+7f_{7,7}(z)]x^6+\ldots$$

Along the aperture centerline B_y is the function $f_{1,1}(z)$. If $f_{1,1}(z)$ is well enough determined the functions $f^{ii}_{1,1}(z)$, $f^{iv}_{1,1}(z)$ and $f^{vi}_{1,1}(z)$ can be computed. A knowledge of B_y along azimuthal lines at $x=x_1, x_2$, etc., can lead to solutions for $f_{3,3}(z)$ and $f_{5,5}(z)$ etc. This would complete the three dipole components.

In practice the higher derivatives of $f_{1,1}(z)$ and $f_{3,3}(z)$, etc. are quite large even after carrying the expansion to an impractical number of terms. However, one knows that this is unphysical and the field components are in fact smooth and continuous.

Express equation (6) in a series of empirical coefficients by grouping in powers of x:

$$[(7)]\ B_y = f_{1,1}(z) + a(z)x^2+b(z)x^4+\ldots$$

where $a(z) = (-1/8)f^{ii}_{1,1}(z) + 3f_{3,3}(z)$

and $b(z) = (1/24\text{x}8)f^{iv}_{1,1}(z)+(-3/16)f^{ii}_{3,3}(z) + 5f_{5,5}(z)$

On the median plane of a dipole magnet of good geometry, the empirical functions a(z), b(z), etc., are small compared to $f_{1,1}(z)$ until well outside into the low fringing fields. One can now express the three components in terms of $f_{1,1}(z)$ and its derivatives and the empirical functions only.

For a window frame dipole, very good accuracy is obtained in stopping at two coefficients.

One obtains:

$$[(8)]\ B_y = [f_{1,1}(z)]+\{[(-y^2\ /\ 2)f^{ii}_{1,1}(z)]+$$
$$+\ [(x^2-y^2)a(z)]\}$$
$$+\ \{[(y^4/24)f^{iv}_{1,1}(z)]+[((-3x^2y^2+y^4)/(6))a^{ii}(z)]$$
$$+\ [(x^4-6x^2y^2+y^4)b(z)]\}$$
$$+\ldots$$

$$[(9)]\ B_x = [(2xy)a(z)]+\{(-xy^3/3)a^{ii}(z)+(4x^3y-4xy^3)\ b(z)\}+\ldots$$

$$[(10)]B_z = [yf^{i}_{1,1}(z)]+\{[(-y^3/6)f^{iii}_{1,1}(z)]$$
$$+\ [((3x^2y-y^3)/((3))a^{i}(z)]\}$$
$$+\ \{[(y^5/24\text{x}5)f^{v}_{1,1}(z)]+[((-5x^2y^3+y^5)/(5\text{x}3\text{x}2))a^{iii}(z)]+$$
$$[((5x^4y-10x^2y^3+y^5)/(5))b(z)]\}+\ldots$$

For a(z) = b(z) = 0, this, of course, reduces to the terms in the [ ] brackets. This is the two dimensional description of the end fringing field.

One can, if they so choose, consider equation [7] as the empirical description of the median plane field and take this as the starting point to calculate the three components elsewhere. However, in the interests of understanding and generality, it seems fruitful to develop the field in terms of the potential multipolarities. These quantities have real conceptual meaning and it is simple to combine them as shown.

Idealized Thick Lens Approximation

The simplest model (apart from a thin lens impulse) is the idealized thick lens with constant two dimensional field properties along the azimuthal direction for a length LB which drop discontinuously to zero at the defined ends. For this square edged model, only the first and second derivative terms in the field expressions will contribute to the beam optical properties. The first derivative gives an angular impulse, and the second derivative a finite displacement, but higher terms contribute nothing. Usually only the first derivative term is used. It is responsible for the simple formula for the vertical focus in the end field of dipoles. The trajectory integration

$$\int_{\text{outside}}^{\text{inside}} f_{n,n}(z)dz$$

is independent of the model to the approximation that the end effect is a thin lens angular impulse. It is clear that the square edged model gives reasonably correct answers for this term, but not nearly as good for the second derivative term.

Our approach at the AGS to facilitate beam transport at the level of this model has been based on precise internal "point" two dimensional and long coil integrating measurements.

From the field equations of Section I, in the end regions there are contributions from the same multipolarities as in the two dimensional interior, but with azimuthally varying coefficients. There are also terms which contain derivatives with respect to Z. At the same azimuthal positions relative to each end of the magnet, these terms are equal and of opposite signs, provided also the transverse positions are the same.

A long integrating coil with its axis parallel to the z axis, and with its ends in regions of zero amplitude of the z derivative terms, responds only to the "two dimensional" allowed multipolarities. The coefficients of these multipolarities, however, are strongly z dependent in the fringing fields and no information other than the azimuthal integral of each term is obtained. The first order term for the axial field B_z is not detected by the long coil, but was shown to be approximated reasonably well by this model.

In the transverse plane, the second derivative term is poorly approximated by this model. The actual effects, however, of the even power terms on trajectories are small and usually some cancellation between the two end effects occurs.

Many computational programs have been developed to trace rays or transfer phase space through linear systems, including the effect of finite chromatic aberrations. Field aberrations are imposed generally in the impulse approximation. (Higher order correction magnets such as sextupoles are ignored in this paper. They are small perturbations and idealized approximations for their properties are quite adequate.)

For well-designed quadrupoles[1] the internal aberrations can be negligible and very insensitive to excitation. The end effects are also quite insensitive to excitation. A quadrupole with four-fold symmetry and a square end on the iron pole has only a significant end effect contribution from the 6θ term. For example a quadrupole with an iron length three times its diameter has about 1/2% aberration due to both ends, at full radius, and falling off as r^4. For shorter quadrupoles this aberration becomes worse, but a quadrupole can be designed so that the 6θ contribution from the ends is compensated inside. Then for any length the beam optical effect of terms other than $f_{2,2}(z)$ and its derivatives can be made very small.

Accepting the limitations inherent in this idealized thick lens model, many beams using general purpose window frame dipoles plus quadrupoles have been set up at the AGS on the basis of these magnetic measurements. Essentially ideal image width transfers are not uncommon.

To go further it requires realistic models for end effects. Knowledge of real aberration terms can be used to influence design of elements. It also permits more accurate trajectory calculations.

There are types of aberrations which can be partially compensated for in a beam, but in general aberrations cause dilution of phase space and physical image broadening whether you can calculate them or not. However, these are applications where a detailed knowledge of the effects of both the above and of chromatic aberrations is absolutely vital. For our purposes the most pressing need for better end field information has been for trajectory calculations in very short window frame dipole magnets used as wide angle spectrometers. An example of such a magnet has been studied. Designated 120D36, it is a window frame magnet of plane geometry, with a 120-in. wide aperture which is 36-in. long and has a 24-in. gap.

Realistic Description of Magnet Fringe Fields

Introduction

To proceed further one must describe the functions $f_{n,n}(z)$ either by calculation or by a series of field measurements as a function of z. Two dimensional calculations using real permeability data can be made. Using the SIBYL[2] program the fringing field of the 120D36 has been calculated.

The x dependence of the field is small. As a result $f_{1,1}(z)$ is very close to its value for an infinitely wide magnet until far out into the weak fringing field. The problem can be solved as a two dimensional array in the yz plane.

Table I shows the excellent agreement between the measured 120D36 $f_{1,1}(z)$ and the computed value. For narrow width apertures considering $f_{1,1}(z)$ to be two dimensional is a somewhat poorer approximation than in this case but the approximation is quite satisfactory for considering the effects of design changes. Using a small cross section exciting coil package suitable for superconducting coils, the effect on $f_{1,1}(z)$ of different excitation arrangements has been computed.

The information used to compute the three components of field as a function of azimuth was obtained by suitable magnetic measurements of the functions $f_{n,n}(z)$.

Field measurements as a function of z and θ at fixed r and some computations of realistic quadrupole end effects have been made but, as described previously, these terms can be made quite small. The dipole field is a more pressing problem and this paper will describe only dipole properties in detail.

Measurements and Analytical Expressions for the Window Frame Dipole Field in Three Dimensions

Equations [8], [9] and [10] give the three field components in a form suitable for precision description of a window frame dipole, requiring three sets of measurements.

On the 120D36 magnet very precise measurements were made using a rigid plane table with an accurate straight edge and with precision displacement blocks in the z direction. The data accuracy is limited by the precision of a nulling potentiometer and by search coil positioning errors. The potentiometer ratios the field against a reference to an accuracy of 1×10^{-4} parts, and is "smooth" to better accuracy. The positional error in the most rapidly changing field region is $\sim 5 \times 10^{-5}$ parts. (For a 6-in aperture magnet this error is about 20×10^{-5} parts. In such a case, a difference pair of coils is used. This reduces appreciably the effect of positional error. The difference data can be converted to a first derivative.)

The vertical components of the magnetic field $f_{1,1}(z)$ measured along the centerline of the magnet can be fitted to an analytical[2] expression of the type

$$f_{1,1}(z) = \frac{B_o}{1 + \ell\, Y(z)}$$

where B_o is the maximum value of $f_{1,1}(z)$ at the center of the magnet, and Y is a polynomial $Y_i = \Sigma_j A_j Z_i^j$.

A computer program calculates the coefficients A_j to fit the values Y_i to the experimental data. The number of data points (about 85) is much larger than the number of degrees of the polynomial. The least squares principle is used to minimize the squared residual. A polynomial of about 23 degrees is used: $\Sigma|R|^2$ being of the order of 1.5×10^{-3}. For a crude calculation there is no need of such a high degree polynomial. The derivatives can be obtained by direct differentiations. The dipolar search coil data when used can be analyzed to give the gradient of the field. Numerical integration gives the field itself before curve fitting to $f_{1,1}(z)$. A second method is to use a Taylor series to expand the experimental values in higher derivatives at the mid-point between the two coils. The linear simultaneous equations for the derivatives are solved by using several consecutive experimental data points. The results of these two methods agree very well.

The values of a (z) and b(z) are obtained by three sets of single coil measurements, one along the centerline and the other two at appropriate different values of x. The solution involves solving two simultaneous equations. As a(z) and b(z) are usually small and slowly varying their derivatives with respect to z may be calculated by the simple difference method. For higher accuracy curve fitting and direct differentiation can be used to obtain their derivatives.

With this data one can generate the three components of the field to high accuracy. This implies that terms in equations (8), (9) and (10) beyond the sixth derivative produce negligible contributions to trajectories. This assumption is directly verified later on.

An alternative to all this one can measure with less accuracy the three components of field at many locations throughout the field and devise a computer program to calculate trajectories through it. For a magnet with very bad symmetries so that many coefficients of the type a(z) and b(z) are necessary, and their amplitudes are large, this is the best procedure. Given a magnet of good symmetries, and with highly developed measurement techniques this is not the case. From a practical point of view the present method was much the easiest way to obtain the data. The function $f_{1,1}(z)$ was measured in a few hours. However, this is only a small part of the advantage.

Method of Trajectory Computations

In rectangular coordinates the equations of motion of a particle with mass m and charge e are

$$[(11)]\quad m\frac{d^2x}{dt^2} = ev_yB_z - ev_zB_y$$

$$[(12)]\quad m\frac{d^2y}{dt^2} = ev_zB_x - ev_xB_x$$

$$[(13)]\quad m\frac{d^2z}{dt^2} = ev_xB_y - ev_yB_x$$

Let α be the angle of a trajectory element ds with respect to the horizontal plane XZ: i.e., the vertical angle. Let θ be the angle of the projection of ds in the horizontal plane measured with respect to the Z axis. The direction of the particle is approximately in the positive Z direction.

By changing variables and using a little manipulation, equations (11), (12) and (13) can be transformed to equations (14) and (15):

$$[(14)]\quad \sin\theta_f - \sin\theta_i = (e/p)\int_i^f(\tan\alpha/\cos\alpha)$$

$$(\sin\theta B_x + \cos\theta B_z)dz - (e/p)\int_i^f(B_y/\cos\alpha)dz.$$

$$[(15)]\quad \sin\alpha_f - \sin\alpha_i = (e/p)\int_i^f(B_x - \tan\theta B_z)dz$$

If the $\sin\theta$-z relation and the $\sin\alpha$-z relations are known the formulas for calculating x and y coordinates are simply

$$[(16)]\quad x_f - x_i = \int_i^f \tan\theta dz$$

$$[(17)]\quad y_f - y_i = \int_i^f (\tan\alpha/\cos\theta)dz$$

Although equations (14) and (15) are too complicated to solve analytically they can be solved numerically by the following scheme.

For the first approximation we calculate the horizontal projection of the trajectory using

$$[(18)]\quad \sin\theta_f - \sin\theta_i = (-e/p)\int_i^f f_{1,1}(z)dz$$

$$[(19)]\quad x_f - x_i = \int_i^f \tan\theta\, dz$$

This is justified as $f_{1,1}(z)$ is much larger than the rest of the terms of the B_y expression and B_y itself is much larger than B_x and B_z. Now we introduce the y-component effect by assuming a projection of the initial values of y and of $\sin\alpha$ through the magnet in the vertical direction. This projected y and α, together with the calculated x and θ values, permits calculation of the three field components B_x, B_y and B_z. Then one makes a first correction to the calculated $\sin\theta$ by using equation [(14)], and to the calculated x-coordinates by using the newly calculated sin θ.

At this stage we can calculate the vertical component sin α by using equation [(15)].

Then calculate the y-coordinate from the expression:

$$[(20)] \quad y_f - y_i = \int_i^f (\tan\alpha/\cos\theta)dz$$

Recalculate the field components, angles and coordinates successively and repeatedly until the elements converge. Actually the convergence is very fast. Only two or three stages of re-calculations are necessary to give a high degree of convergence.

A different approach would be to calculate the trajectory from point to point, generating the three components of field at each z interval. However, the perturbation approach was used so that the effect of different terms in the field expansions could be directly tested. This is very instructive to magnet design and also to trajectory computation.

Results of Trajectory Computations

Very precise trajectories for a given momentum can be calculated for the excitation field at which the measurements were taken. However, the results apply with reasonable accuracy to a wide range of magnet excitation. The proper normalization requires only that

$$\int_{-\infty}^{+\infty} f_{1,1}(z)dz$$

be known as a function of excitation. This is simply obtained with a long integrating coil located on the aperture centerline.

The effect of inserting and removing various terms in the field expressions is informative. The largest effect on the trajectory is the interaction of B_y with the horizontal component of momentum, $p^y \cos\alpha$. The sine of the outgoing horizontal angle, sin θ_f, in particular is very insensitive to details of the trajectory. The other terms in equation [(14)] are very small in their effect. The vertical deflection equation [(15)] is intermediate in strength and quite dependent on details of the trajectories.

For finding the momentum of trajectories the horizontal deflection is by far the most important and easy to measure. For observing magnet optical properties the vertical deflection is very instructive. For the special case of horizontal incidence (sin α_i = 0) we define a vertical focal length as y_f divided by sin α_f. Table II gives an example of the variation of this focal length with y. The final z values are 84-in. from the center of the 120D36 magnet. The near identity of Columns 1 and 2, and of 3 and 4 respectively shows that $f^{iv}_{1,1}(z)$ and $f^{vi}_{1,1}(z)$ which perturb B_y had no effect on the vertical focus. It is also clear from comparing the three orders of B_z that at y = 10-in. the series is not convergent as we are ignoring contributions from still higher terms. However, it should be borne in mind that this is a very short magnet composed entirely of end effects and is also a case with a large 34° deflection. We are dealing with a small perturbation on a term which is itself small. In principle another term $f^{vii}_{1,1}(z)$ could be used. (The term $f^{vi}_{1,1}(z)$ agrees very well for fitting two different polynominal orders to the experimental field data.) However, there seems no point in pursuing this since for momentum analysis any error resulting from this is minute. A similar analysis for an 18D72 magnet with a 6-in. gap shows better vertical convergence with three terms.

Table III illustrates the nonlinearity in focal length as a function of magnet length. (The 18D9 is somewhat artificial, in that a real magnet of this length would have some field deviations from what is found in the ends of a longer magnet.)

For an 18D72, the nonlinearity due to the terms of $f^{iii}_{1,1}(z)$ and $f^{v}_{1,1}(z)$ are smaller than that of $f^{i}_{1,1}(z)$ alone. The latter is caused mainly by the y dependence of B_y itself. Observing Table II again, even for this extreme magnet the nonlinearity in Column 1 using only the term $yf^{i}_{1,1}(z)$ is about one half of Column 5.

The vertical deflection depends on tan θ. Now sin θ_f is very insensitive to the value of y. This appears to support the decoupling of the vertical deflection from the main horizontal term. However, inside the end region itself sin θ is quite y dependent, and this is where it interacts with the vertical deflection. Since the B_y terms are even derivatives, integration through the entire end indeed gives almost zero contribution. This illustrates the value of the present approach. Without the ability to separate effects one would be inclined to attribute the non-linearity more strongly to $f^{iii}_{1,1}(z)$ and $f^{v}_{1,1}(z)$.

The pole ends of dipoles and quadrupoles are sometimes flared to reduce the amplitude of higher order terms in the end fields[4]. Another advantage is that the shape of the field remains more constant with excitation. However, this technique considerably increases the mechanical complexity of a magnet and it is easy to over-estimate the advantages. It is true that saturation of the pole end causes significant shape change with a square edge pole. However, even our shortest transport quadrupoles show length changes of only 12% to pole tip fields of 15 kG and for dipoles 5.6% to 19 kG.

As far as the aberrations due to the higher derivatives are concerned in the case of the quadrupole the influence of 6θ and higher aberrations can be made extremely small. It is true that derivatives of $f_{2,2}(z)$ itself can be made smaller by this flaring, but at the expense of making the end effect regions relatively larger compared to the internal regions.

For the dipole end, the importance of reducing the higher terms in the vertical focus can be misleading, as seen above. It is true that even terms as well as odd will be smaller, however, a significant reduction in $f^{ii}_{1,1}(z)$ and its perturbing of sin θ would increase the fringing field region.

Momentum Calculations

For experimental use of the field description, one wants to extract the momentum for given experimental trajectory coordinates. A first order momentum p_1 can be defined as simply

$$[(21)] \quad p_1 \cos \alpha_i \ (\sin \theta_f - \sin \theta_i) = k \int f_{1,1}(z)dz$$

(Here α_i refers to the initial vertical angle, and θ_f is the final horizontal angle.) A knowledge of the initial and final angles makes this a trivial calculation. In Table IV p_1 is compared with the true momentum p used for calculating widely varying trajectories. With one exception p_1 is accurate to better than 1×10^{-3} parts. The case with $\sim 3 \times 10^{-3}$ parts error is a very extreme ray.

It is obvious that the procedure can be applied easily between any initial and final z values, whether well outside the field or not.

It should be noted that Table IV does not include the effects of a(z) and b(z). To the approximation that this wide angle spectrometer has no x-dependence very accurate momentum can be extracted instantly from a knowledge of the angles. For the experimentally less restrictive problem of the angles and position known at one end and the x-coordinate only known on the other end a similar but less accurate procedure can be set up.

The a(z) and b(z) effects are in practice small in the 120D36, so that at maximum x they contribute about 1%. Over a considerable part of the magnet they can be ignored for quite good precision. However, by treating these effects as perturbations on p_1, and calculating sample trajectories as a function of the parameters, a correction to p_1 can be stored and recalled from a memory of reasonable size.

References

1. G.T. Danby and J.W. Jackson, Proc. Part. Accel. Conf., Wash., D.C., March 1-3 (1967), to be published.
2. The SIBYL program was first written by R.S. Christian.
3. H.A. Enge, Rev. Sci, Instr., 35, 278 (1964).
4. K.G. Steffen, High Energy Beam Optics, (Interscience Publishers, N.Y. 1965).

Table I Comparison of the Calculated and Experimental Values for the Aperture Centerline Field of the 120D36 Magnet.

z [in.]	Calculated F_{11} [kG]	Measured F_{11} [kG]	z [in.]	Calculated F_{11} [kG]	Measured F_{11} [kG]
-18	6.507	6.574	11	2.573	2.575
-14	6.505	6.552	12	2.367	2.370
-10	6.425	6.462	13	2.175	2.178
-8	6.337	6.369	14	1.996	2.000
-6	6.199	6.223	15	1.831	1.835
-4	5.991	6.005	16	1.678	1.682
-3	5.854	5.864	17	1.538	1.541
-2	5.694	5.700	18	1.409	1.412
-1	5.511	5.513	20	1.183	1.186
0	5.304	5.303	22	.996	.997
1	5.077	5.075	24	.841	.840
2	4.833	4.830	26	.713	.710
3	4.577	4.573	30	.520	.512
4	4.313	4.309	34	.388	.376
5	4.045	4.042	38	.296	.281
6	3.780	3.778	42	.232	.213
7	3.519	3.518	50	.151	.129
8	3.266	3.266	58	.107	.082
9	3.023	3.024	62	.081	.054
10	2.792	2.793			

Table II Nonlinearity of Vertical Focal Length of 120D36 as a Function of terms in the Field Expansion.

Y (in)	BY1,BZ1 (in)	BY3,BZ1 (in)	BY2,BZ2 (in)	BY3,BZ2 (in)	BY3,BZ3 (in)
-1	-271.29	-271.29	-271.22	-271.22	-271.22
-3	-269.68	-269.67	-268.98	-268.98	-268.93
-5	-266.51	-266.44	-264.46	-264.46	-264.07
-7	-261.86	-261.59	-257.59	-257.58	-255.97
-10	-252.39	-251.31	-242.68	-242.59	-234.20

NOTE: Data is for total horizontal deflection of about 34° symmetrically about the center of the magnet.

$BY1 = f_{11}(z) - y^2/2 \times f_{11}^{ii}(z)$

$BZ1 = y\, f_{11}^{i}(z)$

$BY2 = BY1 + y^4/24 \times f_{11}^{iv}(z)$

$BZ2 = BZ1 - y^3/6 \times f_{11}^{iii}(z)$

$BY3 = BY2 - y^6/(48 \times 15) \times f_{11}^{vi}$

$BZ3 = BZ2 + y^5/(24 \times 5) \times f_{11}^{v}$

Table III Vertical Focal Length as a Function of Core Length for a 6-in. Aperture Window Frame Magnet.

Y (in)	18D72 (in)	18D36 (in)	18D18 (in)	18D9 (in)
.5	-204.88	-113.12	-68.49	-46.79
1.0	-204.50	-112.78	-68.12	-46.31
2.0	-203.07	-111.05	-66.25	-44.06
2.5	-201.63	-109.25	-64.40	-41.93
3.0	-199.13	-106.18	-61.36	-38.57

Data is for total horizontal deflection of about 34° symmetrically about the center of the magnet.

18D72 Actual magnet 72-in. long
18D36 Field data of 18D72 shortened by 36-in. in the center of the magnet.
18D18 Field data of 18D72 shortened by 54-in. in the center of the magnet.
18D9 Field data of 18D72 shortened by 63-in. in the center of the magnet.

Table IV Comparison for Various Trajectories of a Simple First Order Momentum Calculation with the True Value.

I	II	III	IV	V	VI	VII	VIII	IX	X
p (GeV/c)	X_i (in)	Y_i (in)	Sin θ_i	Sin α_i	X_f (in)	Y_f (in)	Sin θ_f	Sin α_f	p_1 (GeV/c)
1.07234	15.34	9.575	-0.144	-0.061	13.250	-1.195	-0.120	0.0652	1.07184
1.01777	-29.665	11.09	+0.147	-0.068	21.718	-0.685	-0.426	0.0690	1.01715
1.03438	-13.90	-11.805	+0.032	+0.085	15.490	2.807	-0.307	-0.0859	1.03437
1.01777	-32.20	5.880	+0.156	-0.050	20.987	-2.714	-0.434	0.048	1.01779
1.00828	-53.815	7.985	+0.291	-0.049	29.127	-0.716	-0.572	0.048	1.00810
1.01777	-36.475	-15.435	+0.507	+0.105	115.115	5.666	-0.786	-0.092	1.02050
1.06759	8.740	-4.010	-0.132	+0.034	8.824	1.862	-0.133	-0.035	1.06750

Explanation of Column Headings

I. p is momentum used in the trajectory calculation.

II. X_i is initial X coordinate at -50-in. from center of lens.

III. Y_i is initial Y coordinate at -50-in. from center of lens.

IV. Sin θ_i is initial horizontal angle at -50-in. from center of lens.

V. Sin α_i is initial vertical angle at -50-in. from center of lens.

VI. X_f is final X coordinate at +85-in. from center of lens.

VII. Y_f is final Y coordinate at +85-in. from center of lens.

VIII. Sin θ_f is final horizontal angle at +85-in. from center of lens.

IX. Sin α_f is final vertical angle at +85-in. from center of lens.

X. p_1 is the calculated first order momentum.